# THEORETICAL ASPECTS OF THE GENERATION OF RADIO NOISE BY THE PLANET JUPITER

Submitted in fulfilment of the requirements for the degree of Master of Science of Rhodes University

P.A.Deift. July,1971.

Except where it is obvious that I am describing the work of others, the work presented in this thesis is my own.

Percy Deift

## ACKNOWLEDGEMENTS

I would like to thank Professor Gledhill for accepting me as an M.Sc. student and always finding the time to help me. I am particularly grateful for the many, many hours spent in discussing the Jupiter problem with Chris Goertz. Thank you Chris, again and again. Also I would like to thank all the other people in the Rhodes Physics Department who made my stay so pleasant. Lastly, thank you Bagatha, for typing the thesis so quickly and well!

This work was supported by a bursary from the CSIR.

CORRECTIONS

p. 9: Line 21: Instead of "differentition" read "differentiation".

p. 12: Line 22: "...... whenever $\omega$ is, the reduced equation ...."

p. 53: Line 16: "$V_A$ ...... = .32 x $10^8$ m/sec"

$\equiv$ .11c

Lines 21 and 46: "1.23 x $10^8$ m/sec $\equiv$ .41 c ...

.... even for $V_A$ = .41 c equals .91 $\cong$ 1 ....."

p. 54: Instead of ".4c" read ".41c" on the figure.

(Last line) "$\beta^2 = \beta_o^2\ (xo/x)^4$"

p. 55: Lines 16 and 18: Instead of "Ionosphere" read "Iosphere"

Line 25: " $\cong$ 1.09 x $10^5$ km which should ....."

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## INTRODUCTION.

Decameter radiation was first observed from Jupiter by Burke and Franklin (JGR 60, 213, 1955). In 1964 Bigg (Nature, 203, 1008, (1964)) found that Io exerted a profound effect on the radiation.

The majority of the early theories to explain the origin of the decameter emissions, attributed the radiation to an emission process occurring at or near the electron gyrofrequency or the plasma frequency (for a review see eg. Warwick, Space Sci. Rev. 6, 841 (1967)). More recent work centred around the question of how Io modulates the emission (see the article of Carr and Gulkis (Annual Review of Astronomy and Astrophysics Vol 8 (1970)) for a detailed review).

The theories assume either that Io generates the decameter radiation locally (see eg. Gledhill Nature 214, 155, (1967)) or that Io generates a disturbance that propagates through (large) distances in Jupiter's magnetosphere to the source of the decameter radiation, possibly the Jovian Ionosphere (see eg. Goldreich and Lynden-Bell Ap. J. 156 (1969)).

An objection to Gledhill's theory is that there is no apparent source for the high densities required by the model. Goldreich and Lynden-Bell argue that the decameter bursts are due to micro-instabilities initiated by a current of kev electrons flowing along the magnetic flux tube that passes through Io and into the Ionosphere.

A conspicuous success of this theory is the explanation of the conical beaming observed for the decameter radiation, the highly asymmetrical longitude dependence of the bursts, however, (as remarked by Carr and Gulkis (ibid)) is not explained. (See Goertz, PhD Thesis, Rhodes University for a critical discussion of the theory of Goldreich and Lynden-Bell).

Goertz (ibid) takes up an older idea (see eg. Carr and Gulkis, (ibid) p614) that Io generates hydromagnetic disturbances in Jupiter's magnetosphere, which are guided (Alfvén waves) along Io's field line into the Ionosphere: the Alfvén velocity is given by $B_0/(\mu_0\rho)^{\frac{1}{2}}$ (in usual MKS units), so that (see eg. Warwick (ibid)) the waves slow down and steepen (ie. decrease their wavelengths) close in to the denser Ionosphere. This localization of energy can couple to any of a number of instabilities and so generate and/or amplify the decameter radiation.

This thesis considers the transmission of Alfvén waves from Io to the Ionosphere. Various simplified laws for the variation of plasma density are analyzed and juxtaposed to simulate a realistic density variation along the Io-Jupiter flux line. Apparently the Ionosphere is pervious only to high enough frequencies, in excess of 1 Hz. On the other hand the magnetosphere cannot guide the high frequencies efficiently. The Iosphere (that region of the magnetosphere in the vicinity of Io) excercises no containing control over movements faster than $\cong$ .05 Hz.

The analysis is magnetohydrodynamic and both transient and harmonic behaviour is examined.

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## SYMBOLS

The following is a list of (non-standard) symbols used regularly in the text. A definition of the symbol may be found on the page indicated.

| | | |
|---|---|---|
| $\left[\begin{smallmatrix} a_1, a_2; b_1, b_2; \\ c_1, c_2 \end{smallmatrix}\right]$ | p 26 | amplitudes of harmonic waves |
| $a, b, c, a_0$ | p 18 | filter parameters |
| $A, B$ | p 19 | driver parameters; also used (see eg. p 26) for coupling matrices; also used ( see eg. p 40 ) as general (integration) constants |
| $b$ | p 9 | normalised magnetic field: b is also used as a filter parameter, see p 18 |
| $C$ | p 26 | coupling matrix |
| $D_{Io}$ | p 53 | diameter of Io $\cong$ 3000 km |
| $F_1, F_2, F_3$ | p 37 | energy transfer parameters |
| $\bar{H}(s)$ | p 15 | filter transfer function |
| $k$ | p 45 | (generalized) wave number: see also p 57 |
| $k'$ | p 46 | coefficient for standing waves |
| $N, N_{MAX}$ | p 52 | particle densities: also $N_0$, p 54 |
| $p(y)$ | p 15 | $\mathcal{L}^{-1}[\bar{H}(s)]$ |
| $p_0, q_0$ | p 35 | Ionosphere parameters |
| $Q_0$ | p 51 | Ionosphere parameter |
| $r$ | p 30 | filter/transmission parameter: also used(p 52) as the distance from Jupiter's centre |
| $r_0, r_1$ | p 17 | filter parameters: $r_0$ is sometimes used to denote the radius of Jupiter, see eg. p 5 |
| $R_J$ | p 53 | radius of Jupiter $\cong$ 70,000 km. |
| $s$ | p 14 | Laplace variable: also used (p 35) to denote distance |
| $T$ | p 27 | connection matrix: also absolute temperature ( see eg. p 52) |
| $u$ | p 43 | distance variable: also used (p 48) as a characteristic function |
| $v$ | p 9 | normalized velocity |

| | | |
|---|---|---|
| $x$ | p 9 | the coordinate in Cartesian x-direction |
| $x_0, x_1$ | p 13 | filter parameters: the roles of $x_0$ and $x_1$ are sometimes interchanged (see eg. fig.3,p 23: also p 54(ii)) |
| $y$ | p 9 | usually used to denote normalized time: sometimes used to denote Cartesian y-direction eg. p 8 |
| $y_0$ | p 23 | a travel time: $y_0 = \mu_0/2$ |
| $y_x$ | p 24 | a travel time: see also p 41 |
| $\alpha$ | p 31 | filter/transmission parameter |
| $\alpha_0$ | p 19 | filter parameter: also used (see p 40) as a parameter for the law $\beta = \beta_0(x_0/x)^{2/3}$ |
| $\beta$ | p 9 | specific speed: also $\beta_0=\beta(x_0)$(p 26),$\beta_1=\beta(x_1)$(p 16) |
| $\beta^2$ | p 9 | normalized density |
| $\gamma_0$ | p 17 | filter parameter |
| $\delta$ | p 10 | exponent in the density law $\beta=(x_0/x)^\delta$ |
| $\eta_0$ | p 35 | magnetic field ratio |
| $\theta$ | p 31 | filter energy parameter: also used in Section 5,see eg. p 43, as a normalized time: also used as a magnetic colatitude,p 52 |
| $\theta_\xi(x)$ | p 11 | travel time $x_0 \rightarrow x$ |
| $\mu_0$ | p 17 | used in the body of the thesis(Sections 2-5) for the travel $2(\xi_1 - \xi_0)$: used in Sections 1 and 6 as the MKS permeability $\mu_0 = 4\pi \times 10^{-7}$ Henry/m |
| $\xi$ | p 13 | used generally as a length variable: see also p 39 |
| $\xi_0, \xi_1$ | p 13 | filter parameters |
| $\rho_0$ | p 9 | a characteristic plasma density |
| $\tau$ | p 18 | normalized time: see also p 34 |
| $\tau_\xi(x)$ | p 10 | travel time $x_0 \rightarrow x$ |
| $\phi$ | p 31 | used generally as a phase: see also p 48 |
| $\chi$ | p 27 | transmission ratio: also $\chi_T$ p 30 |
| $\nu$ | p 56 | frequency: see also $\nu_i$, p 54 |

## SECTION 1.

## BASIC EQUATIONS

In conventional MKS symbols, the magnetohydrodynamic equations are (see eg. Alfvén and Fälthammar (1963): Cosmical Electrodynamics 2nd Ed. Clarendon Press, Oxford: Alfvén uses CGS units).

(i) $\nabla \times \vec{B} = \mu_0 \vec{J}$

(ii) $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$ with $\nabla . \vec{B} = 0$

(iii) $\rho\, D\vec{v}/Dt = \vec{J} \times \vec{B} - \nabla p + \vec{F}$

(iv) $-\partial \rho/\partial t = \nabla . (\rho \vec{v})$

(v) $p = \text{const}\ \rho^{\gamma}$

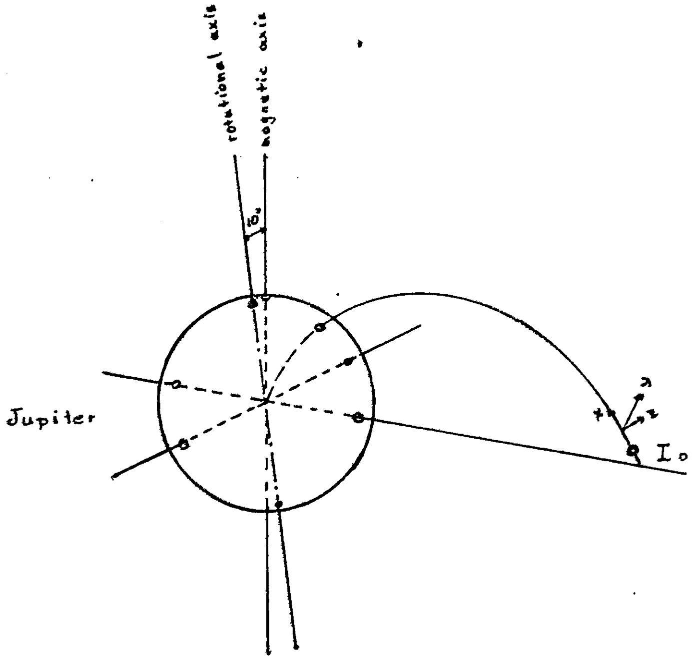
(vi) $\vec{J} = \sigma (\vec{E} + \vec{v} \times \vec{B})$

Denote the above equations (1.1) (i) ... (vi).

Here displacement currents are ignored in Ampère's law (1.1 (i)); "$D/Dt$" is the convective derivative "$\partial/\partial t + \vec{v}.\nabla$" ; $p \propto \rho^{\gamma}$ conserves energy for reversible adiabatic motions; $\sigma$ is the electrical conductivity and $\vec{F}$ is the totality of non-electrical, non-pressure forces acting on a unit volume of plasma.

We will assume that Jupiter's external magnetic field is that of a dipole inclined at an angle of $10^{\circ}$ to the rotational axis. (see eg. Morris and Berge (1962): Astrophys. J. 136 276-282). Io rotates at an L value approximately equal to 6. The corresponding L shell has a radius of curvature in the order of $(6 \times r_0)/2 = (6 \times 70,000)/2 = 210,000$ km where $r_0 = 70,000$ km is Jupiter's radius. Clearly wavelengths much smaller than $2 \times (\pi \times 210,000) \cong 1,200,000$ km will not feel the curvature in the dipole lines. Now we calculate (see section 6) a maximum Alfvén velocity of .46c = .46 x 300,000 = 138,000 km/sec along a field line. With 5 Hz this gives a wavelength 138,000 ÷ 5 = 27,600 km << 1,200,000 km. Thus we are at liberty to straighten out the Io-Jupiter flux line in the analysis. In detail we mark out a Cartesian reference frame, in which the Io-Jupiter flux line unfolds onto the x-axis. The zero of x is in the Iosphere and the Ionosphere is located at both large, positive and large, negative x-values. The y-axis, pointing away from Jupiter, is in a meridian plane of the dipole and the z-axis completes a right-hand system with x and y. Of course the dipole field underlies the x-direction.

We have




The above approximations are particularly suitable in the Iosphere and the Ionosphere, where the total curvature in the field lines is small.

The majority of our considerations will be in this frame of reference unravelled from a spinning dipole. We will make the approximation that the equations (1.1) hold true in the (accelerating) frame. It is understood that the usual rotational forces are included in $\vec{F}$ . The error in Maxwell's equations can be estimated by regarding the acceleration of the frame as due to two effects: the (constant) motion about Jupiter's rotational axis and the $10^{\circ}$ tilt of the magnetic to the rotational axis. The work of Trocheris (Phil Mag. Ser. 7 40 no. 310 Nov 1949 p1143-1154) can be used to show that the former effect is of the order $\Omega D/c$ where $\Omega$ is the angular velocity of the frame, D is a scale of interest and c is the velocity of light. For phenomena on Jupiter ($\Omega$ = 1.76 x $10^{-4}$ rad/sec) influenced by Io (D = 420,000 km = radius of Io's Jovian orbit), we have $\Omega D/c$ = (1.76 x $10^{-4}$ x 420,000)/3 x $10^{5}$ = 2.46 x $10^{-4}$ which is very small compared to 1. Also it is clear that the latter effect can influence only those events less frequent than $\Omega$ . In particular for 5Hz, we have $\Omega$ /5 x (2$\pi$ ) = 1.76 x $10^{-4}$/10$\pi$ = 5.6 x $10^{-6}$ which is again very small compared to 1. The approximation is good!

Now we are including in $\vec{F}$ gravitational, centrifugal and Coriolis forces. (The tilt of the dipole is neglected as an apparent force). The first two are derivable from a potential $\Psi$ (See Gledhill (1967): Goddard Space Flight

Centre Report X-615-67-296) which is unaffected by motions of the Io flux line as measured in the rotating frame of Jupiter.

Conversely we will be able to find motions of the flux line largely independent of $\psi$.

Lehnert (Astrophys. J. (1954) 119 647) measures the ratio of Coriolis force to magnetic force by the parameter $\chi_o = \Omega/\omega$ where $\Omega$ is again the rotational velocity of the planet and $\omega$ is the angular frequency of some hydromagnetic motion. For Jupiter $\Omega = 1.76 \times 10^{-4}$ rad/sec, so that for 5 Hz, $\chi_o = (1.76 \times 10^{-4}) / (2\pi \times 5) = 5.6 \times 10^{-6} \ll 1$. The insignificance of the Coriolis force in the Jovian context affords a considerable simplification as may be seen from the following. The Coriolis force is given by $2\rho(\vec{v} \times \vec{\Omega})$ where $\vec{v}$ is the plasma velocity as measured in the rotating frame and $\vec{\Omega}$ is the angular velocity vector of the planet. For the moment we turn the Cartesian x-axis back into its original dipole, retaining (local) y- and z-directions in an obvious manner. Now clearly motions $\vec{v}$ of the flux tube will couple through $\vec{\Omega}$. At the Iosphere we would expect two circularly polarized characteristic wave modes. At the point approximately half way along the tube between Io and the Ionosphere, where the field direction is in the magnetic equatorial plane, there will be significant coupling from $v_z$ to the longitudinal motion $v_x$. Thus a non-trivial Coriolis force would severely alter the character of waves moving in from Io.

Motions on the scale of Jupiter's radius, however, will be affected. (following Lehnert ibid.)

The preliminary analysis will be for an incompressible, infinitely conducting plasma, assumptions which we will reconsider in a later section. (see Section 6.) Infinite conductivity implies $\vec{E} + (\vec{v} \times \vec{B}) = 0$ with its familiar interpretation of freezing the flux lines into the plasma. Also, we want to investigate $\rho = \rho(x)$, a non-constant function of x, so we interpret incompressibility as $\nabla . (\rho \vec{v}) = 0$ rather than the usual $\nabla . \vec{v} = 0$

With these approximations, and using a vector identity, the equations (1.1) reduce to:

(i) $\partial\vec{B}/\partial t = \nabla\times(\vec{v}\times\vec{B})$

(ii) $\nabla\cdot\vec{B} = 0$

(iii) $\rho\, D\vec{v}/Dt = -\nabla(B^2/2\mu_0 + P + \psi) + (1/\mu_0)(\vec{B}\cdot\nabla)\vec{B}$

(iv) $\nabla\cdot(\rho\vec{v}) = 0$

which we refer to as equations (1.2) ((i) ... (iv)).

We will be interested in solutions to (1.2) such that $\nabla(B^2/2\mu_0 + P + \psi) = 0$. Now as remarked previously, $\psi$ is independent of the motions of the flux tube. Also, we will be regarding $\vec{B}$ as a solution to the remaining equations (1.2). Thus $\nabla P = -\nabla(B^2/2\mu_0 + \psi)$ serves to define P for particular motions in Jupiter's $\psi$-environment. That P does not couple back into the equations is precisely the analytical convenience in assuming incompressibility.

The first results will be appropriate for a region of the magnetosphere, such as the Iosphere, where the underlying magnetic field does not vary significantly. In the Cartesian frame we thus have an underlying $\vec{B} = (B_0,0,0)$ where $B_0$ is a constant. We look for plane x-solutions such that the operators $\partial/\partial y \equiv \partial/\partial z \equiv 0$. Then incompressibility implies that $\rho v_x$ is a constant. At a large distance from a source $v_x$ is zero and so $v_x = 0$ for all x is the consistent solution. $\nabla\cdot\vec{B} = 0$ implies that $B_x$ is a function of time alone. But Faraday's law (1.2) (i) in the x-direction gives $\frac{\partial B_x}{\partial t} = 0$, as $\frac{\partial}{\partial y} \equiv \frac{\partial}{\partial z} \equiv 0$. $\therefore B_x$ = constant = $B_0$. It is easily shown that in the y and z directions Faraday's law also gives $\partial B_y/\partial t = B_0\,\partial v_y/\partial x$, $\partial B_z/\partial t = B_0\,\partial v_z/\partial x$

Momentum conservation (1.2) (iii) gives $\rho\,\partial v_y/\partial t = \frac{B_0}{\mu_0}\,\partial B_y/\partial x$, $\rho\,\partial v_z/\partial t = \frac{B_0}{\mu_0}\,\partial B_z/\partial x$

There is no coupling of the transverse y and z motions, as we expect on physical grounds. We will consider the y-motions

$$\left.\begin{aligned} \partial b_y/\partial t &= B_0\,\partial v_y/\partial x \\ \rho\,\partial v_y/\partial t &= (B_0/\mu_0)\,\partial b_y/\partial x \end{aligned}\right\}\quad (1.3)$$

where we have set $b_y = B_y$. Clearly these equations represent a factorization of the familiar incompressible, perfectly conducting Alvén motions into partial waves. Indeed, we can substitute to obtain $\partial^2 v_y/\partial x^2 = \mu_0\rho/B_0^2\;\partial^2 v_y/\partial t^2 = \frac{1}{V_A^2}\frac{\partial^2 v_y}{\partial t^2}$ where $V_A = B_0/(\mu_0\rho)^{1/2}$ is a (local) Alfvén velocity. We can call such an equation, with varying $\rho$ and hence varying $V_A$, a generalized

Alfvén equation (GAE). The field $b_y$, however, satisfies $\frac{\partial^2}{\partial t^2}(b_y) = \frac{B_0^2}{\mu_0}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial b_y}{\partial x}\right)$ which cannot be reduced to GAE in a non-trivial way. This incompatibility of $b_y$ and $v_y$ motions will in general invalidate such theorems as the equipartition of energy eg. we will see that the kinetic energy $\frac{1}{2}\rho v_y^2$ will not in general equal the magnetostatic energy $\frac{1}{2\mu_0}(b_y^2)$ in the wave. At this point it might be thought that the crucial parameter is $V_A$, giving equal weight to variations in $B_0$ and $\rho^{-1/2}$. That this is not so is seen from (1.3) which remains a good approximation for $B_0 = B_0(x)$, provided the characteristic region of charge for $B_0$ is large compared to a wavelength (see Section 6.). But then $d\xi = dx/B_0(x)$ gives a scale change in which (for $\rho$ a constant) the motions of $b_y$ and $v_y$ are compatible and Alfvén.

Thus through $B_0 = B_0(x)$ we can at most slow down (or speed up) a wave: to change its character we must vary $\rho^{-1/2}$.

We will use normalized variables $v = v_y/(B_0^2/\mu_0\rho_0)^{1/2}$ , $b = b_y/B_0$ , $y = (B_0^2/\mu_0\rho_0)^{1/2}t$ , $\beta = (\rho(x)/\rho_0)^{1/2} > 0$ where $\rho_0$ is a characteristic plasma density. Here $(B_0^2/\mu_0\rho_0)^{1/2}$ is an Alfvén velocity: $v, b$ & $\beta$ are dimensionless while $y$ has units of length.

The equations become
$$\left.\begin{aligned} b_x &= \beta^2 v_y \\ b_y &= v_x \end{aligned}\right\} \quad (1.4),$$

coupling to $v_{xx} - \beta^2 v_{yy} = 0$ where all subscripts indicate partial differentition. We note that $\beta$ is an inverse speed i.e. a normalized time per unit length: we will call it the specific speed. We refer to $\beta^2$, however, as the (normalized) density.

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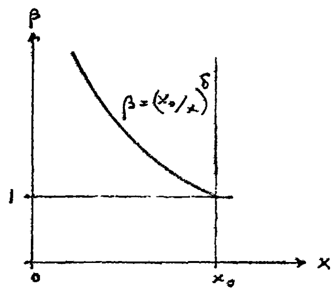
## SECTION 2.

## THE DENSITY LAW $\beta^2 = (\frac{x_0}{x})^4$ (TRANSIENT BEHAVIOUR)

We can make the following classical comments from the theory of partial differential equations about the pair $b_x = \beta^2 v_y$ , $b_y = v_x$ . The characteristics $dy = \pm\beta dx$ imply through Riemann's method (see eg. Sommerfeld: lectures on Theoretical Physics: Vol VI Partial Differential equations in Physics: Academic Press) for hyperbolic equations, a finite communication velocity $\beta^{-1}$ . Now (as in optics) $v_{xx} - \beta^2 v_{yy} = 0$ gives a ray theory only for the higher frequencies, the lower frequencies being denied a $\beta^{-1}$ group mobility. In fact the magnetosphere should whistle at the hydromagnetic frequencies. Also, along a characteristic, we have $db = \pm\beta dv$ : for equipartition of energy we would need $d(b \pm \beta v) = 0$. Thus there is a departition $\pm v d\beta$ in the wave.

In this thesis we will investigate the particular law $\beta = (x_0/x)^\delta$ where $x_0$ is a scale length, for various positive values of the exponent $\delta$. At various stages in the theory, however, and in particular in Section 5, we examine the relevance of phenomena predicted on the basis of the particular law, to a general $\beta$ variation. Before proceeding with this section the reader might find it convenient to refer to Section 6 where the relationship of the law $(x_0/x)^\delta$ to a physically likely density variation along the Io flux line, is discussed.

In $0 < x < x_0$ , then, we have typically



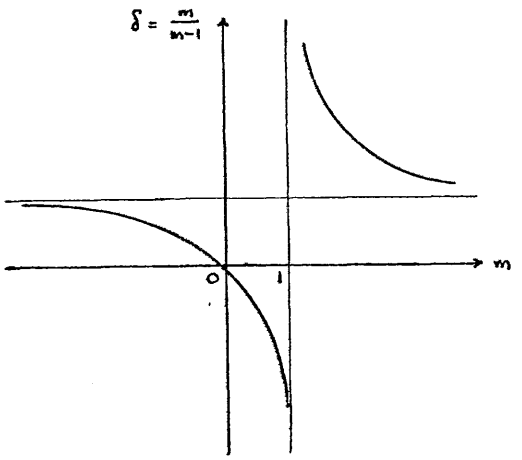

Now $\beta^{-1}$ is a specific speed:hence $\Upsilon_\delta(x) = \int_{x_0}^{x}(-\beta)\,dx$ is a time for signalling from $x_0 \to x < x_0$ . Integrating we obtain for $\delta \neq 1$, $\Upsilon_\delta(x) = \frac{x_0}{1-\delta}\left[1 - (x/x_0)^{1-\delta}\right]$.

Thus if $\delta < 1$ and we imagine $\beta = (\frac{x_0}{x})^\delta$ extends up to $x = 0$ , $\lim_{x\to 0} \Upsilon_\delta(x) = \frac{x_0}{1-\delta}$. But if $\delta > 1$ , $\lim_{x\to 0} \Upsilon_\delta(x) = \infty$ . The law $\delta = 1$ is then the dividing line between laws with finite and infinite travel times $\Upsilon_\delta(0+)$. The physical significance is as follows: when $\delta < 1$ a signal entering $\beta = (x_0/x)^\delta$

from the right (see, in particular, the Ionospheric calculations of Section 3) will reach $x=0$ in finite time, feel the immobility of the (infinitely) dense region $x \gtrsim 0$ , reflect back and set up a standing wave in $0<x<x_0$ . Thus no net energy can be passed into such an (idealized) Ionosphere in a steady state. When $\delta>1$ , however, $\tau_\delta(0+)$ is infinite and the wave never reaches $x=0$ to generate a reflection, and energy can be continuously fed into the Ionosphere from the right.

If we consider the travel time to infinity $\Theta_\delta(x) = \int_{x_0}^{x} \beta\, dx = \frac{x_0}{\delta-1}\left(1-\left(\frac{x}{x_0}\right)^{1-\delta}\right)$ the situation is reversed (we imagine here that $\beta=(x_0/x)^\delta$ extends from $x=x_0$ to $+\infty$ ). Then $\lim_{x\to\infty} \Theta_\delta(x) = \frac{x_0}{\delta-1}$ when $\delta>1$ and $=\infty$ when $\delta<1$ . $\delta=1$ again gives the dividing line.

Evidently it is analytically wise to regard $\delta=1$ as a singularity: one way of accomplishing this, which will prove particularly convenient in the later analysis, is to write $\delta = m/_{m-1}$ for $m \neq 1$ and then $\delta=1$ is obtained only as the $\lim\limits_{m\to\pm\infty} (m/_{m-1})$ . We have

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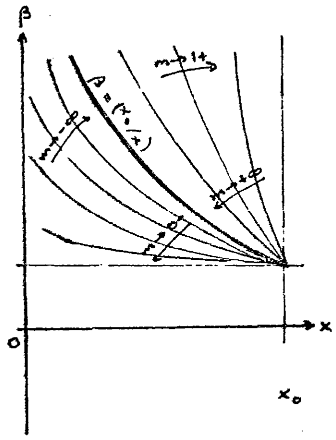



so that $\delta>1$ for $m>1$ and $0<\delta<1$ for $-\infty<m<0$.
Now under a change of variable $\xi = \mp\int\beta\, dx = \pm(m-1)\left[x_0^{(m/_{m-1})} / x^{(1/_{m-1})}\right]$
the GAE $v_{xx} - \beta^2 v_{yy} = 0$ becomes $v_{\xi\xi} - v_{yy} + \frac{m}{\xi} v_\xi = 0,$
which is an important equation in Riemann's unsteady one-dimensional gas dynamics. (see Sommerfeld: Lectures on Theoretical Physics: Vol II: Mechanics of Deformable Bodies: p.265 et seq: Academic Press). Sommerfeld (ibid) references Bechert with a lemma : if $w$ solves $w_{\xi\xi} - w_{yy} + (m/_\xi)\, w_\xi = 0$
then $v = \frac{1}{\xi}\, \partial w/_{\partial\xi} = 2\, \partial w/_{\partial \xi^2}$ solves $v_{\xi\xi} - v_{yy} + (m+2)/_\xi\, v_\xi = 0$
Now the general solution for $m=0$, $v_{\xi\xi} - v_{yy} + \frac{0}{\xi} v_\xi = v_{\xi\xi} - v_{yy} = 0,$

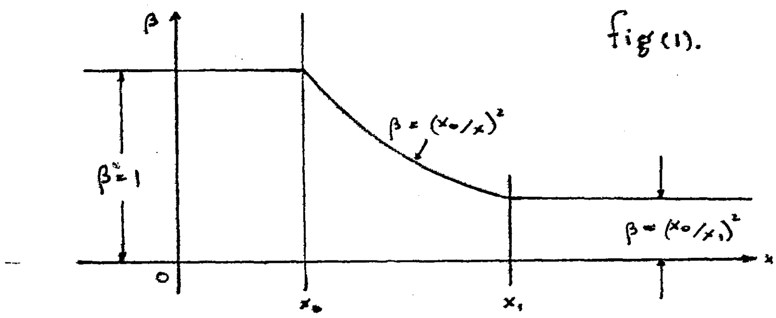
the wave equation, is known: clearly then, by induction, we can obtain the general solution for the densities $m = 0, \pm 2, \pm 4, \ldots$ eg. if $\omega$ solves the wave equation $v = \frac{1}{\xi}\frac{\partial \omega}{\partial \xi}$ solves $m = 2$ : $\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial \omega}{\partial \xi}\right) = \left(2\frac{\partial}{\partial \xi^2}\right)^2 \omega$ solves $m = 4$ : on the other hand $\int \omega \xi \, d\xi = \int \frac{\omega}{2} \, d(\xi)^2$ solves $m = -2$ etc. (We will mention a beautiful connection between the analytic viability of Riemann's equation for $m = 0, \pm 2, \pm 4, \ldots$ and the Bernoulli-Liouville (see eg. Watson: Theory of Bessel Functions p.85 et seq: Cambridge University Press) theory for the solution of Riccati's equation in elementary terms).

The densities $m = 2, 4, \ldots$ give $\delta > 1$ : $m = -2, -4, \ldots$, give $\delta < 1$ . We will now begin a detailed study of the case $m = 2$ ie $\beta = (x_0/x)^2$ or $\beta^2 = (x_0/x)^4$ , the inverse fourth power density law. We remember that as $\delta > 1$ , the travel time $\Upsilon_\delta(0+)$ is infinite. On the other hand $\lim_{x \to \infty} \Theta_\delta(x) = \frac{x_0}{\delta - 1} = x_0$ is finite i.e. there is a finite travel time to infinity. (Note: the finite travel time $\lim_{x \to \infty} \Theta_\delta(x) = x_0$ arises because the Alfvén velocity $\beta^{-1}$ gets arbitrarily large. This will imply a signalling velocity greater than the speed of light, which is impossible. The detail is that in $\beta^2 v_y = b_x$ we have conserved momentum non-relativistically. Clearly we must be circumspect in using the physics $v_{xx} - \beta^2 v_{yy} = 0$ ). Also $\xi = x_0^2/x$ (where we have chosen the plus sign).

Now as $\frac{\partial \omega}{\partial \xi}$ is a solution of the wave equation $\omega_{\xi\xi} - \omega_{yy} = 0$ whenever $\omega$ is the reduced equation $v_{\xi\xi} - v_{yy} + 2v_\xi/\xi = 0$ has a general solution $\frac{1}{\xi} f(y + \xi) + \frac{1}{\xi} g(y - \xi)$ where $f$ & $g$ are arbitrary except for some obvious mathematical requirements.

Then the GAE $v_{xx} - \beta^2 v_{yy} = v_{xx} - (x_0/x)^4 v_{yy} = 0$ has a general solution $x f(y + x_0^2/x) + x g(y - x_0^2/x)$ . For $x > 0$ the wave $x f(y + x_0^2/x)$ moves and increases to the right: $x g(y - x_0^2/x)$ moves and decreases to the left. Also we notice that the solution $xf + xg$ to the GAE would be given by a WKB method: $x_0^2/x$ , as a primitive of the specific speed $\beta = (x_0^2/x^2)$ is the generalized phase, while $x$, inversely proportional to a fourth root of $\beta^2 = (x_0/x)^4$ , is the WKB growth factor. We might say then that $\beta^2 = (x_0/x)^4$ is accurately a WKB medium. Of course, for a high frequency ray theory we have $v \propto \Upsilon^{-1/4}$ ( & $b \propto \Upsilon^{1/4}$ ) for any density law $\Upsilon$ (as obtained by Alfvén and Fälthammar (ibid) p.87)

We consider an Iospheric variation of $\beta$ as in fig (1).




For $x \le x_0$, $\beta = 1$. For $x \geq x_1$, $\beta = \beta_1 = (x_0/x_1)^2$, a constant $< 1$. For $x_0 \le x \le x_1$, $\beta = (x_0/x)^2$. Io generates somewhere to the left of $x_0$. We will sometimes refer to the region $x_0 \to x_1$ as the filter. For convenience we change scale as follows:

| | | | |
|---|---|---|---|
| for | $x \le x_0$ | set | $\xi = x - 2x_0$ |
| | $x_0 \le x \le x_1$ | set | $\xi = -x_0^2/x$ |
| | $x_1 \le x$ | set | $\xi = \beta_1 x - 2\beta_1 x_1$ |

The variable $\xi$ increases continuously with $x$ through $x_0$ and $x_1$. At $x_0$, $\xi = \xi_0 = -x_0$. At $x_1$, $\xi = \xi_1 = -x_0^2/x_1$. Also we have $\xi_0 < \xi_1 < 0$.

The preceding remarks on the speed $(x_0/x)^2$ mean that the general solution to the GAE in $x_0 \to x_1$, is given by $\frac{\omega}{\xi}$ where $\omega$ is the general solution to the wave equation $\omega_{\xi\xi} - \omega_{yy} = 0$

Thus we must solve the equations

$$\left.\begin{array}{lll} v_{\xi\xi} - v_{yy} = 0 & \text{in} & \xi \le \xi_0 \\ \omega_{\xi\xi} - \omega_{yy} = 0 & \text{in} & \xi_0 \le \xi \le \xi_1 \\ v_{\xi\xi} - v_{yy} = 0 & \text{in} & \xi_1 \le \xi \end{array}\right] \quad (2.1)$$

The equations couple at the boundaries $x_0$ & $x_1$ where we demand continuity of $v$ and $v_x$, conditions which are conveniently obtained from the equations $v_x = b_y$ and $\beta^2 v_y = v_x$ using Feynman's method (see Feynman, Lectures on Physics, Vol II, Chapter 33-3). In detail, the method would give us continuity of $v$ and $b$. But if the necessary limiting operations are interchangeable - a condition we will certainly assume - then $v_x = b_y$ gives the continuity of $v_x$. While the condition on $v$ may be regarded as obvious for connected bodies, that on b precludes the existence of surface currents at $x_0$ or $x_1$ .

In the normalized variables the boundary conditions become

$$\left.\begin{array}{lll} \text{at } x_0 & v(\xi_0^-, y) = \frac{1}{\xi_0}\, w(\xi_0^+, y) & \\ & v(\xi_0^-, y) = -{}^1/_{\xi_0^2}\, w(\xi_0^+, y) + {}^1/_{\xi_0}\, w_\xi(\xi_0^+, y) & \\ \text{and at } x_1 & ({}^1/_{\xi_1})\, w(\xi_1^-, y) = v(\xi_1^+, y) & \\ & -({}^1/_{\xi_1^2})\, w(\xi_1^-, y) + (\tfrac{1}{\xi_1})\, w_\xi(\xi_1^-, y) = v_\xi(\xi_1^+, y) & \end{array}\right\} \quad (2.2)$$

Now we assume that Io generates a wave $f(\xi - y)$ travelling to the right in $\xi \le \xi_0$ . The wave reaches $\xi = \xi_0$ at time $y = 0$ so that $f(\xi) = 0$ for $\xi > \xi_0$ but as we include shock waves in the analysis $f(\xi_0)$ is not generally assumed equal to zero. Where necessary, the reader should interpret in the sense of generalized functions. We will say that the driver $f$ decays if $\lim_{\xi \to -\infty} f(\xi)$ exists and equals zero. This latter condition will be assumed where appropriate in the theory.

We will use the Laplace Transform method for its ease in incorporating initial conditions. Also at this stage we can mention that a non-trivial convergence problem arises in the analysis which seems to have a natural resolution in the Laplace Transform method. At a later stage it is convenient to adopt a more direct, physical approach. In what follows all attempts at mathematical rigour are abandoned. There should be no difficulty in providing the justification for the method. As in all transform techniques this is probably best given a posteriori.

The equation $v_{\xi\xi} - v_{yy} = 0$ for $\xi \le \xi_0$ transforms in time to $\frac{d^2\bar{v}}{d\xi^2} - s^2\bar{v} = -s\,v(\xi, 0+) - v_y(\xi, 0+)$ where $\bar{v}$ is the transform of $v$ , and $s > 0$ is the Laplace variable.

But $v(\xi, 0+) = f(\xi)$ and $v_y(\xi, 0+) = -f'(\xi)$ .

$\therefore \frac{d^2\bar{v}}{d\xi^2} - s^2\bar{v} = f'(\xi) - s\,f(\xi)$ , which has a general solution $\bar{v} = e^{-s\xi}\int_{\xi_0}^{\xi} e^{s\mu} f(\mu)\,d\mu + A\,e^{-s(\xi-\xi_0)} + B\,e^{s(\xi-\xi_0)}$ , where $A$ and $B$ are functions of $s$ yet to be determined. Let us set $\bar{f} = \int_0^\infty e^{-sv} f(\xi_0 - v)\,dv$ . Now $\bar{v}(\xi)$ is bounded as $\xi \to -\infty$ , so we must have $A = -\bar{f}$ . Then where $v_0 = \bar{v}(0)$, we get $\bar{v}(\xi) = e^{-s\xi}\int_{\xi_0}^{\xi} e^{s\mu} f(\mu)\,d\mu + \bar{f}\,e^{-s(\xi-\xi_0)} + (v_0 - \bar{f})e^{s(\xi-\xi_0)}$ for $\xi \le \xi_0$ .

In the region $\xi_0 \to \xi_1$, $w_{\xi\xi} - w_{yy} = 0$ transforms to $\frac{d^2\bar{w}}{d\xi^2} - s^2\bar{w} = 0$

which solves to $\bar{w} = \left[\dfrac{w_0 e^{-s\xi_0} - w_1 e^{-s\xi_1}}{e^{-2s\xi_0} - e^{-2s\xi_1}}\right] e^{-s\xi} + \left[\dfrac{w_0 e^{s\xi_0} - w_1 e^{s\xi_1}}{e^{2s\xi_0} - e^{2s\xi_1}}\right] e^{s\xi}$

where $\bar{w}$ is the transform of $w$ and $w_0 = \bar{w}(\xi_0)$ , $w_1 = \bar{w}(\xi_1)$.
For $\xi \geq \xi_1$, $v_{\xi\xi} - v_{yy} = 0$ transforms to $\frac{d^2}{d\xi^2}(\bar{v}) - s^2\bar{v} = 0$, giving
$\bar{v} = Ae^{-s(\xi-\xi_1)} + Be^{s(\xi-\xi_1)}$ where $A, B$ are again constants. Here the boundedness of $\bar{v}$ as $\xi$ increases implies $B = 0$ and where $v_1 = \bar{v}(\xi_1)$, we get $\bar{v} = v_1 e^{-s(\xi-\xi_1)}$

We refer to the totality of expressions for $\bar{v}$ and $\bar{w}$ in the three regions as (2.3). The boundary conditions (2.2) transform to

$$\left.\begin{array}{lll} v_0 = w_0/\xi_0 & \& & \bar{v}'(\xi_0) = -1/\xi_0^2\, w_0 + \bar{w}'(\xi_0) \\ v_1 = w_1/\xi_1 & \& & -1/\xi_1^2\, w_1 + \frac{1}{\xi_1}\bar{w}'(\xi_1) = \bar{v}'(\xi_1) \end{array}\right\} \quad (2.4)$$

It is now a matter of algebra to substitute (2.3) into the four equations (2.4) and solve for the four unknowns $v_0, v_1, w_0, w_1$.
Eventually we obtain $v_1 = -(2\xi_0)^2 \, (s\bar{f}) \, \bar{H} \, e^{-s(\xi_1-\xi_0)}$ ——— (2.5)

where $\bar{H} = \bar{H}(s) = s\Big/\left(1 + 2s(\xi_0 - \xi_1) - 4s^2\xi_1\xi_0 - e^{-2s(\xi_1-\xi_0)}\right)$

The factor $(\xi_1 - \xi_0)$ in (2.5) is the time for a pulse to move from $x_0$ to $x_1$. This explains the delay $e^{-s(\xi_1-\xi_0)}$ in the response at $x_1$ to a driver at $x_0$. Now $\bar{f}$ would give the field at $x_0$ if there were no filter $x_0 \to x_1$ and $\beta = 1$ for all $x$. Then $s\bar{f}$ would give its derivative. (It is true but not trivial that $s\bar{f}$ gives the transform of the derivative of $f$ in the theory of generalized functions as well.) In anticipation of later results we choose to regard this product $s\bar{f}$ as the systems driver, rather than $\bar{f}$ alone. The implication is that the Iosphere is a generalized A.C. device. Following engineering usage, $\bar{H}(s)$ is called the system transfer function. Let $p(y)$ be the inverse transform $p(y) = \mathcal{L}^{-1}\{\bar{H}(s)\}$.

In investigating $p(y)$ we will apply the initial and final value theorems of Laplace Transform Theory to $\bar{H}(s)$. These state, respectively, that under suitable conditions $\lim_{y\to 0+} p(y) = \lim_{s\to\infty} s\,\bar{H}(s)$ and $\lim_{y\to\infty} p(y) = \lim_{s\to 0} s\,H(s)$.

We must emphasize that the reader interested in rigorizing the theory would justify the applicability of these two theorems, particularly the latter, as a very central result.

At a later stage we will consider an already-mentioned and related convergence problem. In what follows let us measure zero time from $y = (\xi_1 - \xi_0)$.

Now for the shock $f(\xi - y) = \mathcal{U}((\xi_0 - \xi) - y)$ (where $\mathcal{U}$ is the unit Heaviside), $\bar{f} = \int_0^\infty e^{-sv} f(\xi_0 - v)\,dv = \int_0^\infty e^{-sv}\,\mathcal{U}(v)\,dv = 1/s \quad \therefore s\bar{f} = 1$. Thus $\bar{H}$ is, within a factor, the response of the system to a shock input. We have

$$1 + 2s(\xi_0 - \xi_1) - 6s^2\xi_1\xi_0 - e^{-2s(\xi_1 - \xi_0)} = -4s^2\xi_1\xi_0 - 2s^2(\xi_1 - \xi_0)^2 + o(s^3)$$

$$= -2s^2(\xi_1^2 + \xi_0^2) + o(s^3) .$$ Hence $\lim_{y \to \infty} p(y) = -1/2(\xi_0^2 + \xi_1^2)$ .

Also $\lim_{y \to 0+} p(y) = -\frac{1}{4\xi_1\xi_0}$ .

Thus for a shock input, $\lim_{y \to \infty} v(\xi_1, y) = 2\xi_0^2/(\xi_1^2 + \xi_0^2) = 2/1 + \beta_1$ and $\lim_{y \to 0+} v(\xi_1, y) = \xi_0/\xi_1 = \frac{x_1}{x_0}$. When $\beta_1 = 1$ i.e. $x_0 = x_1$ and there is no filter, $\lim_{y \to \infty} v(\xi_1, y) = \lim_{y \to 0+} v(\xi_1, y) = 1$ , which we certainly expect. When $x_1 >> x_0$, $\beta_1$ is small so that $v(\xi_1, y) \sim 2$ for large $y$ . The physical picture is that of two <u>non-growing</u> waves of amplitude $\sim +1$ moving in opposite directions in the filter $x_0 \to x_1$, coupling to the unit driver and a reflected wave of amplitude $\sim \mp 1$ at $x_0$, and coupling to a transmitted wave of amplitude $\sim 2$ at some distant point $x_1$ . Clearly no net energy passes the point $x_0$ . We begin to form the idea that it is difficult for energy to escape continuously from the Iosphere, at least at the lower frequencies. To interpret $\lim_{y \to 0+} v(\xi_1, y) = x_1/x_0$ we remember that the shock $\mathcal{U}$ triggers a wave $x\, f(y + x^2/2)$ moving to the right in the filter from $x_0$ . At $y = 0+$ (in the original time scale) we must have $x_0 f(x_0) = 1$ to match boundary conditions. The phase $x_0$ in $f$ will reach $x_1$ at time $y = \xi_1 - \xi_0$ . Thus at time $y = (\xi_1 - \xi_0)$, the zero of the shifted scale, we have $v(\xi_1) = x_1 f(x_0) = x_1 (\frac{1}{x_0}) = \frac{x_1}{x_0}$ . The larger $x_1$, the more the wave can grow in its passage from $x_0$. The physics in the growth is quite elementary: towards $x_1$ the medium is getting lighter. To balance the forces in the wave front, the decrease in inertia implies larger velocities. On the other hand, a $v$-wave moving from $x_1$ to $x_0$ , will decrease.

In systems engineering, $p(y)$ is sometimes called the weighting function or the memory function. This is because $v(\xi_1, y)$ is proportional to the convolution of the driver $s\bar{f}$ with $\bar{H}(s) = \mathcal{L}\{p(y)\}$ . It follows that events in the driver $\frac{d}{dy} f(\xi_0 - y) = \mathcal{L}^{-1}\{s\bar{f}\}$ which occur at a time $y$ can be recalled to an extent $p(y' - y)$ at a time $(y' - y)$ later. Hence memory function! The significance of the result $\lim_{y\to\infty} p(y) = -\tfrac{1}{2}(\xi_1^2+\xi_0^2) = \text{const} \neq 0$ is that the system has infinite memory. Moreover, it remembers remote events equally. If everything of significance in the driver occurs in finite time, so that $\frac{d}{dy}[f(\xi_0-y)]$ becomes small, then the system will eventually recall each event equally and it remembers them in a simple sum. Now physically, events are changes $\frac{d}{dy}[f(\xi_0-y)]$ in the pulse so that only a driver with net variation can permanently affect the Iosphere. Thus a pulse with $f(\xi_0) = \lim_{\xi\to-\infty} f(\xi) = 0$ will eventually be forgotten, while a shock, with its interpretation as a generalized function, will be remembered. Apparently, then, very low frequency motions can sustain convective movements perpendicular to the field lines, particularly as $v$ increases WKB away from the Iosphere. Such motions might produce an ex-Iospheric trigger for Sonnerup and Laird's (see JGR (1963) <u>60</u> no 1 pp 131-139) interchange instability.

We initiate now a systematic investigation of

$$\bar{H}(s) = s/((1+2\xi_0 s)(1-2\xi_1 s) - e^{-\mu_0 s}) \quad , \qquad \text{where } \mu_0 = 2(\xi_1-\xi_0) > 0$$

(<u>Note</u>: do not confuse this $\mu_0$ with the permeability $\mu_0$ of free space; the latter has been normalized out of the equations in (1.4)), is twice the travel time for the filter $x_0 \to x_1$. In particular we will interpret it as the time for a signal to leave $x_1$, be reflected from $x_0$ and return to $x_1$. If we take $s$ large enough we can expand $\bar{H}(s)$ as

$$\bar{H}(s) = s \sum_{k=0}^{\infty} \gamma_0^{k+1} \frac{e^{-\mu_0 k s}}{(s+r_0)^{k+1}(s-r_1)^{k+1}}$$

where $r_1 = 1/2\xi_1 < 0$, $r_0 = 1/2\xi_0 < 0$, $\gamma_0 = -1/4\xi_1\xi_0 = -r_1 r_0 < 0$

As $\xi_0 < \xi_1 < 0$, $r_1 < r_0 < 0$.

If the filter were to extend beyond $x_1$, so that $\beta = (x_0/x)^2$ for all $x \geq x_0$, then $-2\xi_0 = 2x_0$ would be the time required for a signal from $x_0$ to be reflected from infinity and return to $x_0$ (see previous discussion on finite travel times). Similarly $-2\xi_1 = 2x_0^2/x_1$ gives the time for signalling and return to $x_1$. Thus $-r_0 = -1/2\xi_0$ $(-r_1 = -1/2\xi_1)$ gives a natural frequency for the filter $x_0 \to \infty$ $(x_1 \to \infty)$. Of course $|r_1| > |r_0|$. Clearly these ideas tie in with the above interpretation of $\mu_0$.

Each of the terms $k$ in $\bar{H}(s)$ represents a wave arriving at $x_1$ at a (shifted) time $\mu_0 k$. The waves are generated by successive reflections back and forth between $x_0$ and $x_1$, as is familiar in optical filter theory. The first wave is proportional to $s/(s+r_0)(s-r_1)$, the second wave creates an impression which is a factor $\gamma_0/(s+r_0)(s-r_1)$ of the first, and so on. Now $\gamma_0/(s+r_0)(s-r_1) = \left[\frac{\gamma_0}{-(r_0+r_1)}\right]\left[\frac{1}{s+r_0} - \frac{1}{s-r_1}\right]$ has an inverse Laplace transform $\frac{1}{2(\gamma_0+\gamma_1)}\left[e^{-r_0 y} - e^{r_1 y}\right]$. This function $\frac{1}{2(\gamma_0+\gamma_1)}\left[e^{-r_0 y} - e^{r_1 y}\right]$ is a typical element in the total memory of the system: the element operates from one wave to the next. We can now understand the mechanism of the total recall implied by $\lim_{y\to\infty} p(y) = \text{constant} \neq 0$ : as $r_0 < 0$, $e^{-r_0 y}$ increases with $y$ and so remembers the distant past. On the other hand $e^{r_1 y}$ will emphasize recent events.

A corollary of the infinite memory $e^{-r_0 y}$ is that it is not obvious that the behaviour of the system is convergent. The problem comes into relief when we analogize the facter $\frac{\gamma_0}{(s+r_0)(s-r_1)}$ as a mechanical system. With a mass $m_0$, a friction $\nu_0$, and a spring constant $k_0$ we would require $k_0/m_0 = -r_1 r_0 = \gamma_0 < 0$ and $\frac{-\nu_0}{m_0} = r_1 - r_0 < 0$ . Thus the analogy requires $k_0 < 0$ a negative spring constant, giving an unstable system. Thus there is no local physics to give us the intuition for stability.

Apparently stability must come from the co-operative behaviour of the entire filter. We see that as $\gamma_0$ is negative in $\frac{\gamma_0}{(s+r_0)(s-r_1)}$ the wave $k$ will oppose the effect of the wave $(k-1)$ etc. A term $e^{-r_0 y}$ will, on reflection, generate waves proportional to $-e^{-r_0(y-\mu_0)}$ and $-(y-\mu_0)\, e^{-r_0(y-\mu_0)}$ (amongst others). The non-trivial problem (see Appendix) is that all these waves converge for large $y$ . Physically we may expect the Iosphere to act as an underdamped system with overshoot.

In the appendix we show (essentially) that we can write

$$\begin{aligned} p(y) &= p(\tau) \\ &= \gamma_0 \sum_{k=0}^{[\tau]-1} \frac{(-a)^{k+1}}{(k+1)!} e^{b(\tau-k-1)} \left[c(\tau-k-1)\right]^{k+2} \left\{ f_k[c(\tau-k-1)] - f_{k+1}[c(\tau-k-1)] \right\} \\ &\quad + \gamma_0\, e^{r_1 \mu_0 \tau} \\ &\quad + (-r_0 a_0) \sum_{k=0}^{[\tau]} \frac{(-a)^k}{k!} e^{b(\tau-k)} \left[c(\tau-k)\right]^{k+1} f_k[c(\tau-k)] \end{aligned} \qquad (2.6)$$

where $0 < a < \frac{2r_1 r_0}{(r_1+r_0)^2} < \frac{1}{2}$, $b = -\frac{(r_1-r_0)^2}{2r_1 r_0} < 0$, $c = \frac{r_1^2 - r_0^2}{2r_1 r_0} > 0$, $a_0 = \frac{2r_1 r_0}{r_1+r_0} < 0$

$\tau$ is a normalized time $\tau = \frac{y}{\mu_0}$ and the notation $[\tau]$ gives the largest integer $\leq \tau$. The first term in (2.6) is defined as zero for $\tau = 0$. The functions $f_k$ are modified spherical Bessel functions of the first kind. We have $f_k(z) = \left(\frac{\pi}{2z}\right)^{1/2} I_{k+\frac{1}{2}}(z)$ . These functions can be expressed in terms of elementary functions eg. $f_0(z) = \sinh(z)/(z)$ (see Appendix).

For $\tau \leq 1$ we get

$$p(\tau) = \left[\frac{-r_1 r_0}{r_1 + r_0}\right]\left[r_1 e^{r_1\mu_0\tau} + r_0 e^{-r_0\mu_0\tau}\right] \text{ giving } \quad p(0) = -r_1 r_0 = \delta_0 ,$$

giving, for a shock input, $\lim_{y\to 0+} v(\xi_1, y) = -(2\xi_0)^{-1}\gamma_0 = \frac{x_1}{x_0}$, which we have obtained previously.

It is possible to prove the following facts about $-p(\tau)$ for $\tau \leq 1$. Certainly $-p(\tau)$ is positive in the range. It has a maximum value $r_1 r_0$ at $\tau = 0$ and a minimum value $(r_1 r_0)\alpha_0^{\frac{1-\alpha_0}{1+\alpha_0}}$ (where $\alpha_0 = \frac{r_1}{r_0} = \frac{x_1}{x_0} > 1$ ) at

$\tau = [1/\mu_0(r_1+r_0)] \ln (r_0/r_1)^2 < 1$. Also we have from the final value theorem $\lim_{y\to\infty} [-p(y)] = (r_1 r_0)(2r_1 r_0/r_1^2 + r_0^2) = \frac{2 r_1 r_0}{\alpha_0 + \frac{1}{\alpha_0}}$. We can then establish the following inequalities for the maximum and minimum values: $(r_1 r_0)\alpha_0^{(1-\alpha_0/1+\alpha_0)} <$
$< (r_1 r_0)\frac{2}{\alpha_0 + \frac{1}{\alpha_0}} < r_1 r_0$, which gives the overshoot. Also we have

$$-p(\tau = 1) = \left[\frac{\alpha_0 e^{1-\alpha_0} + e^{1-\alpha_0^{-1}}}{1+\alpha_0}\right](r_1 r_0) .$$

We have, of course $\quad (r_1 r_0) \; \alpha_0^{(1-\alpha_0)/(1+\alpha_0)} < (r_1 r_0) \; \dfrac{\alpha_0 e^{1-\alpha_0} + e^{1-\alpha_0^{-1}}}{1+\alpha_0} < (r_1 r_0)$.

We can draw a sketch

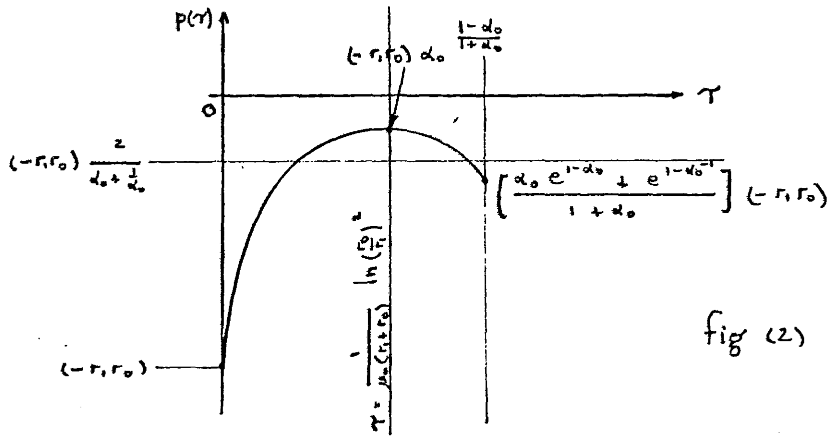



fig (2)

where we remark that $(r_1 r_0)\,\frac{2}{\alpha_0 + \frac{1}{\alpha_0}}$ is not necessarily less than $\left[\frac{\alpha_0 e^{1-\alpha_0} + e^{1-\alpha_0^{-1}}}{1+\alpha_0}\right](r_1 r_0)$

Graph (1) plots $p(\tau) = \mathcal{L}^{-1}\{\bar{H}(s)\}$ for various values of $r_0$ & $r_1$. The properties of $p(\tau)$ mentioned above for $\tau \leq 1$ are displayed in the curves. Unless $\alpha_0 = \frac{r_1}{r_0} = \frac{x_1}{x_0}$ is large, there is very little oscillation in the response for $\tau > 1$ ie. the driver is cancelled almost immediately by the first reflection.

The physics is as follows: When the travel time $\mu_0$ is small, which occurs for a given $x_0$ when $x_1 \cong x_0$, the reflection $\kappa = 1$ contains as up-to-date, though inverted, image of the wave $\kappa = 0$ and cancellation will be complete: when

$\mu_0$ is large ( $x_1 >> x_0$ for the given $x_0$) significant events can occur in $k=0$ of which $k=1$ has no immediate cognizance; the time $\mu_0$ for information to be communicated becomes important and it takes that much longer for a steady state to be achieved. Clearly we can extend these ideas to show that the filter will track accurately any motions in the driver $s\bar{f}$ which are slower than $\mu_0$.

Another view of the problem is that perturbations should take a time $\mu_0$ to die out.

In Graph (1) we see that the asymptotes of $-p(\tau)$ increase with $|r_1|$ : this is to be expected for a given driver acting on progressively lighter media ($\frac{1}{10} < \frac{1}{5} < \frac{1}{2}$). (In reconverting to $y = \mu_0 \tau$ it should be remembered that $\mu_0$ is different for each of the three curves.)

Graph (2) shows the response of the filter to a driver $f(\xi_0 - y) = Ay\, e^{-By}$ The numbers attached to the curves are $r_0, r_1, B, A$, respectively: thus eg. -1,-2,.5 , 1 gives the response of an Iosphere $r_0 = -1$, $r_1 = -2$ to a driver $y\, e^{-.5y}$. To see the desteepening of the waves we must normalize the driver $y\, e^{-.5y}$ by the specific speed $\beta_1$ at $x_1$ for each Iosphere $r_0, r_1$.

The dotted line on the Graph (2) represents this normalization for the case $r_0 = -1$, $r_1 = -2$. The growth of the response with respect to the driver is of course the WKB effect.

Graph (3) shows the response of the filter to a driver $f(\xi_0 - y) = A \sin By$ The first four numbers attached to each curve are as in the preceding paragraph: the fifth number is the time interval used in the numerical integrations. Clearly the filter is capacitive, letting in the higher frequencies. This may be regarded as evidence for a more general coupling theory which we will consider under the harmonic analysis ( see Section 4). Also, as expected, it does not take many oscillations to reach a steady state. Graph (4) gives the response to the single pulse $f(\xi_0 - y) = [1 - \mathcal{U}(y - \frac{\pi}{B})] \times$ $\times [A \sin By]$, where $\mathcal{U}$ is the unit Heaviside. The numbers attached to the curves are as in Graph (3).

We defer a full $r_0, r_1$ - parametric analysis of $p(y)$ to Section (4).

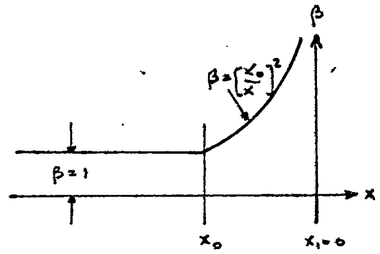
———————

## SECTION 3

## THE DENSITY LAW $\beta^2 = (\frac{x_o}{x})^4$ (TRANSIENT BEHAVIOUR) (CONTD)

If we displace the driver in Section 2 from the high to the low density terminal of $\beta = (x_o/x)^2$ and reverse its direction, we move from a consideration of the Iosphere to a consideration of the Ionosphere. Clearly much of the physics, particularly after a time $\mu_o$ and into the steady state, will be the same. In the total problem, however, there are features specific to the Ionosphere. In this section we will briefly discuss two of them.

Firstly, in the Ionosphere, it is sometimes permissible to neglect internal reflections. For example, the decameter instability (which gives rise to the observed radiation in Goertz's theory) might eat up the pulse before it can reach and be reflected from the lower Ionosphere. Or perhaps visc‡ous effects at the lower Ionosphere/upper Atmosphere are severe enough to damp out the return wave. The convenient picture is clearly




With $x_1 = 0$ the travel time $\mu_o$ (given by $2x_o(1 - \frac{x_o}{x_1})$ when $x_1 \neq 0$ ) is infinite so that we have formally precluded reflections off $x_1$ . (note that here $x_o < 0$ : compare with Iosphere).

Now consider a driver $v = f_o(y-x)$ moving in from the left and arriving at $x_o$ at time $y = 0$ . There will be a reflected wave $f_1(y+x)$ back into $x < x_o$ and a transmitted wave $x\, g(\frac{x_o^2}{x} + y)$ into the Ionosphere. In analogy with a pulse moving along an increasingly heavy string, $x\, g(\frac{x_o^2}{x} + y)$ propagates (slower and slower) to the right, diminishing WKB in size with the distance.

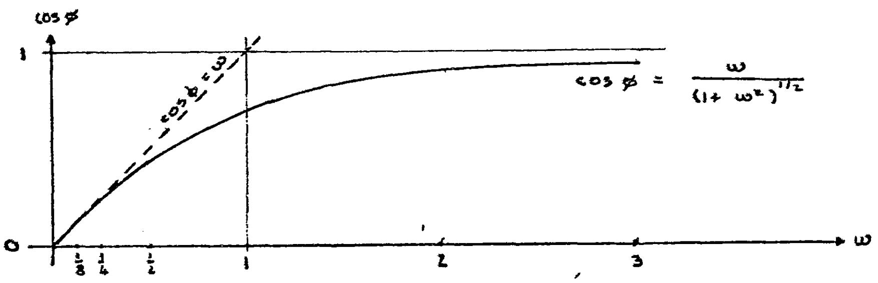
If we require continuity of $v$ and $v_x$ at $x = x_o$ we get

$$\left.\begin{aligned} f_o(y-x_o) + f_1(y+x_o) &= x_o\, g_o(x_o+y) \\ -f_o'(y-x_o) + f_1'(y+x_o) &= g_o(x_o+y) - x_o\, g_o'(x_o+y) \end{aligned}\right] \quad (3.1)$$

solving to

$$x_o\, g_o(x_o+y) = e^{y/2x_o} \int_0^y \frac{d}{dy}\{f_o(y-x_o)\}\, e^{-y/2x_o}\, dy \quad (3.2)$$

which gives an explicit physics. The filter responds only to variations $\frac{d}{dy}(f_0(y-x_0))$ in the driver, as emphasized previously. If $\frac{d}{dy}(f_0(y-x_0))$ becomes small, the integral in (3.2) ceases to change and the factor $e^{y/2x_0}$ reduces ($x_0 < 0$!) the field $x_0 g_0(x_0+y)$ at $x_0$ to zero. Thus for a shock $f_0(y-x_0) = \mathcal{U}(y)$ ($\mathcal{U}$ the unit Heaviside) we have $x_0\, g_0(x_0+y) = e^{y/2x_0} \xrightarrow[y]{\infty} 0$. On the other hand we anticipate capacitive behaviour. Indeed for a sinusoid $f_0(y-x_0) = \sin(\omega y)$ we get $x_0\, g_0(x_0+y) = \cos\phi \sin(\omega y + \phi) - \sin\phi \cos\phi\, e^{y/2x_0}$ where $\phi = \arccos \frac{\omega}{[\omega^2 + (1/2x_0)^2]^{\frac{1}{2}}}$ and $0 < \phi < \pi/2$. As $\cos\phi$ increases with the frequency $\omega$, the capacitive behaviour is apparent. Moreover the behaviour is critical around $\omega = 0$ as may be seen (for $\frac{1}{2x_0} = 1$) from




The relevant physics is in the basic equation $v_x = b_y$. At steady state, or at low frequency, $b_y \cong 0$. Thus $v_x \cong 0$. But the heavy region near $x_1 = 0$ is immobile. Hence $v \cong 0$ for all $x$. It is wonderful how the mathematics compacts this physics into the formula $x\, g_0(x_0^2/x + y)$ : from the second equation (3.1) we see that in a steady state $g_0(x_0+y)$ can only be zero.

Finally we remark that the decay $e^{y/2x_0}$ from $\mathcal{U}(y)$ in the Ionosphere is in contrast to the initially growing response $e^{-r_0 y}$ to a similar shock in the Iosphere (see Section 2). The reader will find that detailed consideration of this point gives an additional, non-trivial insight into the workings of the filter.

The second feature specific to the Ionosphere is the uncertainty in boundary conditions. As with the mathematical theory for the Earth, there are difficulties with the plasma condition at the lower Ionosphere. Nevertheless there is likely to be a rather narrow region of precipitous decrease in plasma density towards the upper atmosphere. In a picture

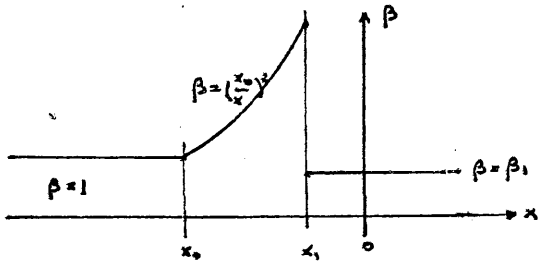



fig (3)

we can model this effect, be it crudely, by disconnecting $\beta_1$ from $(x_0/x)^2$ and making it small.

There is the following important physical consequence: the convected velocity field $v$ enlarges into the light medium $\beta_1$ . But the electric field $E_z$ is given by $v_y B_0$ (see Section 1) proportional to $v$ : thus, basically to conserve energy, the Poynting term $E_z b$ makes $b$ small in the region $x>x_1$ . We have then that $x_1$ is a significant reflection point for $b$ -field and there is accumulation in $x<x_0$. This is obviously of importance for an instability which relies for its basic physics on Fermi type collisions between particles and magnetic field.

We could analyze the Ionosphere in fig. (3) using the Laplace Transform method of Section 2. There are additional technical difficulties due to the disconnection at $x_1$ which fragments the modified spherical Bessel functions $f_\kappa$ . It is particularly apt to say that we would be concerned with a theory of functions which bear a derivative relationship to the $f_\kappa$'s , as the associated Legendre functions $P_l^m(z)$ to the functions $P_l(z)$ . We prefer, however to look directly at the waves, as in the previous paragraphs. The method is straightforward: the wave $v = x\, g_0(x_0^2/x + y)$ reaches $x_1$ at a time $y_0 = x_0(1-\frac{x_0}{x_1})$ transmitting the wave $h_0(y-\beta_1 x)$ into $x>x_1$ and reflecting the wave $x\, g_1(y-\frac{x_0^2}{x})$ back into $x<x_1$ . Again matching boundary conditions we get

$$x_1 g_1(y - x_0^2/x_1)$$
$$= e^{-y(\beta_1 x_1 + x_0^2/x_1)^{-1}} \int_{y_0}^{y} e^{y(\beta_1 x_1 + x_0^2/x_1)^{-1}} \left\{ \frac{(\frac{x_0^2}{x_1} - \beta_1 x_1)}{((\frac{x_0}{x_1})^2 + \beta_1)} g_0'(\frac{x_0^2}{x_1} + y) - \frac{g_0(\frac{x_0^2}{x_1} + y)}{((\frac{x_0}{x_1})^2 + \beta_1)} \right\} dy$$

At a time $2y_0$ this wave will reach $x_0$ and so on. (From our experience in the Iosphere we expect the method to converge).

The Graphs (5) give a dynamic portrayal of the passage of a driver

$$v = f_0(y - x_0)\sin(2\omega y)$$

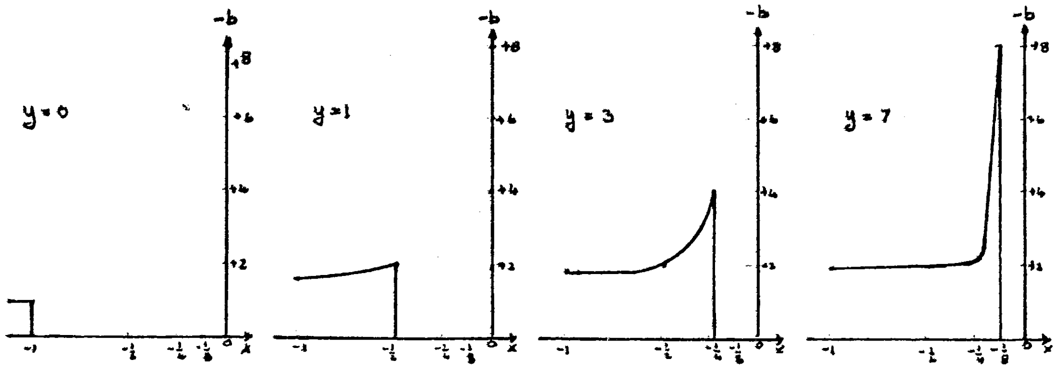
into an Ionosphere $x_0 = -\frac{1}{4}$, $x_1 = -\frac{1}{8}$. Also we have set $\beta_1 = 1$ : as $(\frac{x_0}{x_1})^2 = [\frac{(-1/4)}{(-1/8)}]^2 = 4$, this constitutes a disconnection of 3.

The curves plot $b$-field as a function of distance in the filter $x_0 \to x_1$, at successive time intervals of approximately $y = \frac{1}{4} y_0 = \frac{1}{4}((-\frac{1}{4}) - (-\frac{1}{4})^2/(-\frac{1}{8}))$ $= \frac{1}{4}(\frac{1}{4}) = \cdot 0625$. The $b$ field increases towards the higher densities at $x_1 = -\frac{1}{8} = -\cdot 125$, as mentioned briefly in Section 2. ($b$ is calculated from the above method as $b(y) = \int_0^y v_x \, dy$.)
Alfvén and Fälthammar (loc cit) observe that the increase is basically a statement of the conservation of energy. In addition, the waves steepen towards $x_1$ ie. the wavelength decreases and there is a localization of energy in space. It is at points of the medium where the localization has been sufficient to accumulate a large field energy per unit particle, that instabilities will be initiated.

We remind the reader that the frequency of the pulse in the medium is that of the driver: there is a change only of wave number, not of frequency. The product $\rho v$ is a non-linearity in space, not in time!

An important feature is the development of nodes and antinodes (at $x \simeq -\cdot 25, \simeq -\cdot 17, \simeq -\cdot 125$ and $x \simeq -\cdot 2, \simeq -\cdot 14$ resp.) An instability which feeds off $b$ should have hot spots approximately at the antinodes. It is meaningless to extend a magnetohydrodynamic analysis beyond this point.

Lastly we mention that the phenomenon-orientated physicist should not be too hasty in dismissing the low frequencies as uninteresting. This is because every real pulse is finite and contains a non-trivial spectrum of frequencies in the front and the tail. Though the body of the wave be at low frequency and without incident, the onset (and decay) of the wave should register as an Ionospheric event. The extreme case is the shock $\mathcal{U}(y)$ : the response $e^{y/2x_0}$ travels from $x_0$ as the pair $v = (\frac{x}{x_0}) e^{\frac{1}{2x_0}(y-y_x)}$ and $b = (2 - \frac{x_0}{x}) e^{\frac{(y-y_x)}{2x_0}} - 2$ where $y_x = x_0(1 - \frac{x_0}{x})$ and $y - y_x$ is thus the travel time from $x = x_0$ to $x = x_1$. In the wave front (i.e. $y = y_x$), $b(x, y_x) = -x_0/x$, giving accurately a WKB increase. Also there is the associated steepening/localization. Thus we should see at successive times, (with $x_0 = -1$):




One must expect, as mentioned previously, that close enough to $x = 0$, $b$ couples to an instability. Movements at low frequency must not be dismissed, a priori, as influencing the decameter radiation.

Also low frequencies reveal a fundamental difference between the characteristic motions of the filter and those of its surround. In the medium $x < x_0$, a motion $v = f_0(y-x) = \mathcal{U}[y-(x-x_0)]$ is possible: but across the boundary at $x_0$, the wave feels the increasing inertia of the medium and such $v$-motions are progressively damped. Thus in $x > x_0$ only motions with $v_x < 0$ are possible. The flux law $v_x = b_y$ then gives $b_y = 0$ for $x < x_0$, but $b_y \neq 0$ for $x > x_0$. Thus in $x < x_0$ the plasma can move steadily and maintain a constant tilt $\arctan b = \arctan (b_x/B_0)$, say, in the field lines: in $x > x_0$, however, plasma motion will continuously stretch the lines. Verily, then, the flux in $x > x_0$ is anchored at heavy $x \lesssim 0$ while that in $x < x_0$, if uncoupled from $x_0$, is free.

We mention that the Ionospheric flux lines, stretching out from the immobile base $x \approx 0$ to gain the tension to reduce the motions $v$, eventually assume a non-trivial tilt $= \arctan b = \arctan \lim_{y \to \infty} \left[ \left(2 - \frac{x_0}{x}\right) e^{(y-y_x)/2x_0} - 2 \right] = -\arctan 2$.

Detailed consideration of this point for a polarization $b_z$ gives an explanation of the "lead" of Io's effect at Jupiter from its dipole flux tube (see Goertz and Deift, to be published: will be referenced in Goertz, PhD thesis, Rhodes University).

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## SECTION 4

### THE DENSITY LAW $\beta^2 = \left(\frac{x_0}{x}\right)^4$ (HARMONIC BEHAVIOUR).

In this section we look for harmonic solutions in a filter.

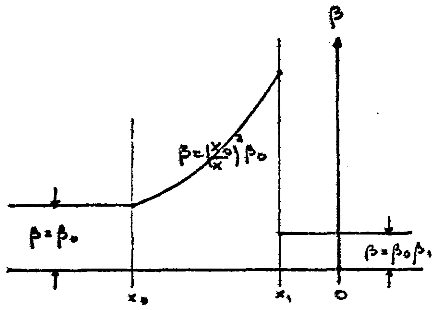



fig (4)

, where $\beta_0 = \left[\frac{\gamma(x_0)}{\rho_0}\right]^{1/2}$ is not necessarily 1.

We assume that in

$$x \le x_0 \,, \quad v = a_1 e^{i\omega(y - \beta_0 x)} + a_2 e^{i\omega(y + \beta_0 x)} \qquad \text{with} \quad b = -a_1\beta_0 e^{i\omega(y - \beta_0 x)} + a_2 \beta_0 e^{i\omega(y + \beta_0 x)}$$

$$x_0 \le x \le x_1 \,, \quad v = b_1 x e^{i\omega(y + \beta_0 x_0^2/x)} + b_2 x e^{i\omega(y - \beta_0 x_0^2/x)} \qquad \text{with} \quad b = -b_1\left(\frac{\beta_0 x_0^2}{x} + \frac{i}{\omega}\right) e^{i\omega\left(y + \frac{\beta_0 x_0^2}{x}\right)} + b_2\left(\frac{\beta_0 x_0^2}{x} - \frac{i}{\omega}\right) e^{i\omega\left(y - \frac{\beta_0 x_0^2}{x}\right)}$$

$$x_1 \le x \,, \quad v = c_1 e^{i\omega(y - \beta_0\beta_1 x)} + c_2 e^{i\omega(y + \beta_0\beta_1 x)} \qquad \text{with} \quad b = -c_1\beta_1\beta_0 e^{i\omega(y - \beta_1\beta_0 x)} + c_2\beta_1\beta_0 e^{i\omega(y + \beta_1\beta_0 x)}$$

(4.1)

and examine the coupling at $x_0$ and $x_1$ . (The actual field is obtained, as usual, by taking real parts). The reader will notice that we use amplitudes "$a$" in $x \le x_0$, "$b$" in $x_0 \le x \le x_1$, "$c$" in "$x_1 \le x$" , respectively: also, a subscript "1" indicates travel to the right, "2" to the left. We require $\omega \neq 0$ .

Coupling is obtained, as always, through continuity in $v$ and $v_x$ .

At $x_0$ we then obtain $A a = B_0 b$

where $a = \binom{a_1}{a_2}$ , $b = \binom{b_1}{b_2}$

$$A = \begin{bmatrix} e^{-i\omega\beta_0 x_0} & e^{i\omega\beta_0 x_0} \\ (-i\omega\beta_0) e^{-i\omega\beta_0 x_0} & (i\omega\beta_0) e^{i\omega\beta_0 x_0} \end{bmatrix} , \quad B_0 = \begin{bmatrix} x_0 e^{i\omega\beta_0 x_0} & x_0 e^{-i\omega\beta_0 x_0} \\ (1 - i\omega\beta_0 x_0) e^{i\omega\beta_0 x_0} & (1 + i\omega\beta_0 x_0) e^{-i\omega\beta_0 x_0} \end{bmatrix}$$

$$\& \quad A^{-1} = \left[\frac{1}{2i\omega\beta_0}\right] \begin{bmatrix} (i\omega\beta_0) e^{i\omega\beta_0 x_0} & -e^{i\omega\beta_0 x_0} \\ (i\omega\beta_0) e^{-i\omega\beta_0 x_0} & e^{-i\omega\beta_0 x_0} \end{bmatrix} , \quad B_0^{-1} = \frac{1}{(2i\omega\beta_0 x_0^2)} \begin{bmatrix} (1 + i\omega\beta_0 x_0) e^{-i\omega\beta_0 x_0} & -x_0 e^{-i\omega\beta_0 x_0} \\ (-1 + i\omega\beta_0 x_0) e^{i\omega\beta_0 x_0} & x_0 e^{i\omega\beta_0 x_0} \end{bmatrix}$$

We denote all the formulae by (4.2)

At $x_1$ we obtain $B_1 b = C c$ where $c = \binom{c_1}{c_2}$ , (again) $b = \binom{b_1}{b_2}$ ,

$$B_1 = \begin{bmatrix} x_1\, e^{i\omega\beta_0 x_0^2/x_1} & x_1\, e^{-i\omega\beta_0 x_0^2/x_1} \\ (1 - i\omega\beta_0 x_0^2/x_1)\, e^{i\omega\frac{\beta_0 x_0^2}{x_1}} & (1 + i\omega\beta_0 x_0^2/x_1)\, e^{-i\omega\beta_0 x_0^2/x_1} \end{bmatrix}, \quad C = \begin{bmatrix} e^{-i\omega\beta_1\beta_0 x_1} & e^{i\omega\beta_1\beta_0 x_1} \\ (-i\omega\beta_1\beta_0)\, e^{-i\omega\beta_1\beta_0 x_1} & i\omega\beta_1\beta_0\, e^{i\omega\beta_1\beta_0 x_1} \end{bmatrix}$$

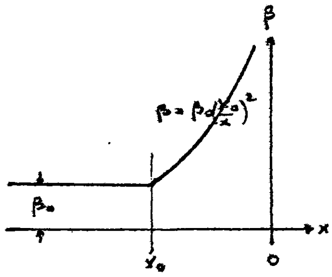
and

$$B_1^{-1} = \frac{1}{(2 i\omega\beta_0 x_0^2)} \begin{bmatrix} (1 + i\omega\beta_0 x_0^2/x_1)\, e^{-i\omega\frac{\beta_0 x_0^2}{x_1}} & -x_1\, e^{-i\omega\frac{\beta_0 x_0^2}{x_1}} \\ (-1 + i\omega\beta_0 x_0^2/x_1)\, e^{i\omega\frac{\beta_0 x_0^2}{x_1}} & x_1\, e^{i\omega\frac{\beta_0 x_0^2}{x_1}} \end{bmatrix}, \quad C^{-1} = \frac{1}{(2 i\omega\beta_0\beta_1)} \begin{bmatrix} i\omega\beta_0\beta_1\, e^{i\omega\beta_0\beta_1 x_1} & -e^{i\omega\beta_0\beta_1 x_1} \\ i\omega\beta_0\beta_1\, e^{-i\omega\beta_0\beta_1 x_1} & e^{-i\omega\beta_0\beta_1 x_1} \end{bmatrix}$$

We denote all these formulae by (4.3).

The total filter performance is given by $c = Ta$ where $T = C^{-1} B_1 B_0^{-1} A$. $T$ may be termed the connection matrix. Clearly we have from (4.2), (4.3) that $T^{-1}$ exists for $\omega \neq 0$. Hence we have that the only solution for $Ta = 0$ (or $T^{-1}c = 0$) is the trivial $a = 0$ (or $c = 0$). Thus a driver $a_1 \neq 0$ will always leak through the filter: conversely the filter is never a perfect magnetohydrodynamic mirror. Also, the invertibility of $T$ makes it impossible to set up a standing wave in the filter by driving it only from one end.

Consider firstly an Ionosphere




We drive from $x < x_0$ $(a_1 = 1)$ and there are no sources in the Ionosphere $(b_2 = 0)$. Then from (4.2) the transmission problem, in particular, requires the solution of $A\binom{1}{a_2} = B_0\binom{b_1}{0}$ for $b_1$.

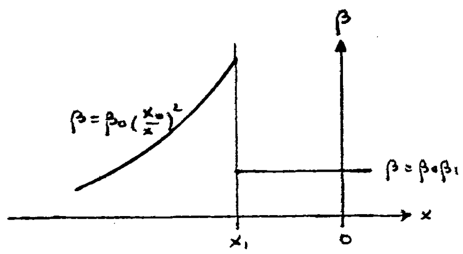
We get $b_1 \times e^{i\omega(y + \beta_0 x_0^2/x)} = e^{i\omega(y - \beta_0 x_0)} \left( \frac{2 i\omega\beta_0 x_0}{-1 + 2 i\omega\beta_0 x_0} \right)$ at $x_0$. The transmission ratio $= \chi =$ transmitted wave/driver $= 2i\omega\beta_0 x_0 / {-1 + 2i\omega\beta_0 x_0} = \left[ \omega\beta_0 / \left[ \left(\frac{1}{2x_0}\right)^2 + (\omega\beta_0)^2 \right]^{\frac{1}{2}} \right] e^{i\phi}$ (where $0 < \phi = \arccos\left[ \omega\beta_0 / \left[ (\omega\beta_0)^2 + (1/2x_0)^2 \right]^{\frac{1}{2}} \right] < \frac{\pi}{2}$), which, for $\beta_0 = 1$ checks with a formula (and a graph $|\chi|$) in Section 3. We can understand the high frequency bias, specifically as it appears in the coupling problem, as follows: $A\binom{1}{a_2} = B_0\binom{b_1}{0}$ expands to

$$e^{-i\omega\beta_0 x_0} + a_2\, e^{i\omega\beta_0 x_0} = b_1 x_0\, e^{i\omega\beta_0 x_0}$$

$$(-i\omega\beta_0)\, e^{-i\omega\beta_0 x_0} + (i\omega\beta_0)\, a_2\, e^{i\omega\beta_0 x_0} = (1 - i\omega\beta_0 x_0)\, e^{i\omega\beta_0 x_0}\, b_1 .$$

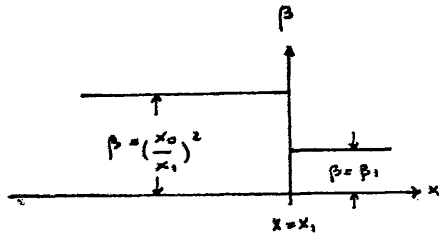
For good transmission $a_2 \simeq 0$. But then $a_1 e^{-i\omega\beta_0 x_0} \cong b_1 x_0 e^{i\omega\beta_0 x_0}$ and $(-i\omega\beta_0) a_1 e^{-i\omega\beta_0 x_0} \simeq (1 - i\omega\beta_0 x_0) b_1 e^{i\omega\beta_0 x_0}$ must be compatible. This is only possible if $\beta_0\omega \gg \frac{1}{x_0}$. Now $\beta_0\omega$ is the wave number of the driver to the left of $x_0$. But in Section 3 we showed that a shock (with $v_x = 0$ ) is damped $(v_x < 0)$ on crossing into the Ionosphere : moreover, for a given $\beta_0$, this generation of wave-number is proportional to $\frac{1}{x_0}$. Thus the inequality $\frac{1}{x_0} \ll \beta_0\omega$ will be true if the additional curvature in the pulse as it slows down in the Ionosphere, is small compared to the wave-number in the driver: dynamically, as $v_x = b_y$ the inequality makes the motions $b_y$ compatible across the boundary and there is no need to excite other motions, like the reflected wave, to restore continuity to the physics. We get good transmission! Conversely at low frequency (and hence small wave number) the effect of the medium predominates generating motions $b_y$ in the Ionosphere entirely different from the driver: these motions can only be matched in a strong reflection. In terms of lengths we say that only those wavelengths $\frac{1}{\beta_0\omega}$ which are much smaller than the Ionosphere (length $x_0$ ), are transmitted.

With this behaviour we should contrast the coupling problem from the lower Ionosphere into the upper Atmosphere




With $b_1 = 1$ and $c_2 = 0$ we must solve $B_1 \binom{1}{b_2} = C \binom{c_1}{0}$ for $c_1$ to get $c_1 e^{i\omega(y - \beta_1 x)} = \left[\frac{2i\omega\beta_0 x_0^2/x_1}{1 + i\omega\beta_0(\beta_1 x_1 + x_0^2/x_1)}\right]\left[x_1 e^{i\omega(y + x_0^2\beta_0/x_1)}\right]$ at $x = x_1$.

The transmission ratio $\mathcal{X}$ equals $(2i\omega\beta_0 x_0^2/x_1)/(1 + i\omega\beta_0(\beta_1 x_1 + x_0^2/x_1)) \xrightarrow[\omega]{\infty}$ $2/[1+\beta_1(\frac{x_1}{x_0})^2]$ which is equal to 1 only when $\beta_1 = (\frac{x_0}{x_1})^2$ ie. when the density is connected at $x_1$. The reader will recognize $2/(1+\beta_1(\frac{x_1}{x_0})^2)$ as the transmission ratio for the non-dispersive filter (see eg. Alfvén and Fälthammar, loc cit, p85 et seq).




The physics is obvious from the preceding paragraph: to get through a filter unmodified, a pulse must have significant variation in a characteristic length of the medium. But here the meduim changes in zero distance. Clearly the disconnection introduces an irremovable incompatibility across $x_1$. At best, high frequency in the ioncsphere can remove the variation in $x<x_0$ (and in $x>x_0$, if any) : a reflection $1-\chi = 1-\dfrac{2}{1+\beta_1(\frac{x_1}{x_0})^2}$ from $x_0$, however, always remains.

Macroscopically, the coupling problem appears as the off-diagonal terms in the connection matrix. If the incompatibility of motions across the boundary is removable the off-diagonal terms in the matrix should become small with frequency.

Thus at $x_0$, $A^{-1}B_0 = \begin{bmatrix} e^{2i\omega\beta_0 x_0}\left(\frac{-1}{2i\omega\beta_0}+x_0\right) & \left(\frac{-1}{2i\omega\beta_0}\right) \\ \left(\frac{1}{2i\omega\beta_0}\right) & e^{-2i\omega\beta_0 x_0}\left(\frac{1}{2i\omega\beta_0}+x_0\right)\end{bmatrix} \underset{\omega}{\overset{\infty}{\sim}}$

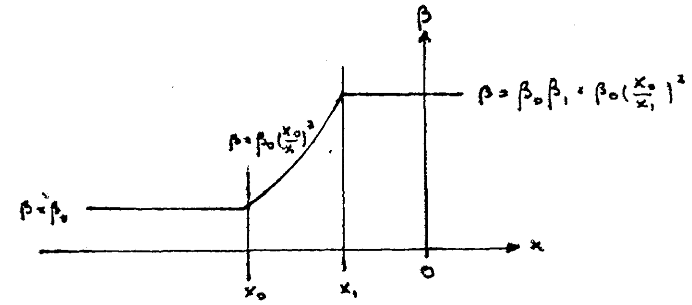
$\begin{bmatrix} e^{2i\omega\beta_0 x_0}x_0 & 0 \\ 0 & e^{-2i\omega\beta_0 x_0}x_0\end{bmatrix}$. But at $x_1$, $B_1^{-1}C =$

$$= \begin{bmatrix} e^{-i\omega\beta_0(\beta_1 x_1 + x_0^2/x_1)}\left(\dfrac{1+i\omega\beta_0(\beta_1 x_1+x_0^2/x_1)}{2i\omega\beta_0 x_0^2}\right) & e^{i\omega\beta_0(\beta_1 x_1 - \frac{x_0^2}{x_1})}\left(\dfrac{1-i\omega\beta_0(\beta_1 x_1 - \frac{x_0^2}{x_1})}{2i\omega\beta_0 x_0^2}\right) \\ e^{-i\omega\beta_0(\beta_1 x_1 - x_0^2/x_1)}\left(\dfrac{-1-i\omega\beta_0(\beta_1 x_1 - x_0^2/x_1)}{2i\omega\beta_0 x_0^2}\right) & e^{i\omega\beta_0(\beta_1 x_1 + x_0^2/x_1)}\left(\dfrac{-1+i\omega\beta_0(\beta_1 x_1 + x_0^2/x_1)}{2i\omega\beta_0 x_0^2}\right)\end{bmatrix}$$

$$\underset{\omega}{\overset{\infty}{\sim}} \begin{bmatrix} e^{-i\omega\beta_0(\beta_1 x_1 + x_0^2/x_1)}\left(\dfrac{\beta_1 x_1 + x_0^2/x_1}{2x_0^2}\right) & e^{i\omega\beta_0(\beta_1 x_1 - x_0^2/x_1)}\left(\dfrac{-(\beta_1 x_1 - x_0^2/x_1)}{2x_0^2}\right) \\ e^{-i\omega\beta_0(\beta_1 x_1 - x_0^2/x_1)}\left(\dfrac{-(\beta_1 x_1 - x_0^2/x_1)}{2x_0^2}\right) & e^{i\omega\beta_0(\beta_1 x_1 + x_0^2/x_1)}\left(\dfrac{(\beta_1 x_1 + x_0^2/x_1)}{2x_0^2}\right)\end{bmatrix}$$

Only when $\beta_1 x_1 = x_0^2/x_1$ (ie. $\beta_1 = (x_0/x_1)^2$ ) does the incompatibility disappear. But then the medium is reconnected at $x_1$.

The above results have been concerned with coupling per se. Additional features arise when we allow two coupling points to interfere, as in the filter, fig (4). It is sufficient to consider a reconnected filter, driven from the right




as in the Iosphere. If we solve $\binom{c_1}{1} = T\binom{0}{a_2}$ for $a_2$, we obtain

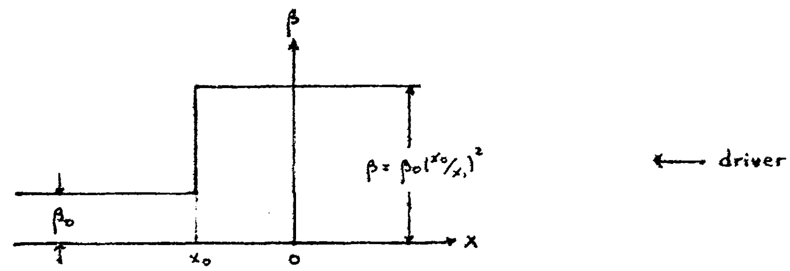
$$a_2\, e^{i\omega(y+\beta_o x)} = \left[(2\omega\beta_o x_o^2/x_1)^2 \Big/ \left[(1-2i\omega\beta_o x_o)(1+2i\omega\beta_o x_o^2/x_1) - e^{-2i\omega\beta_o y_o}\right]\right] e^{i\omega(y+\beta_1 x_1)}\, e^{i\omega(\beta_1 x_1 - \beta_o x_o)}$$

at $x_o$

giving a transmission ratio $\chi_T = \left[(2\omega\beta_o x_o^2/x_1)^2 \Big/ (1-2i\omega\beta_o x_o)(1+2i\omega\beta_o x_o^2/x_1) - e^{-2i\omega\beta_o y_o}\right] e^{-iy_o\beta_o\omega}$

As discussed in Section 2, $\beta_o y_o = \beta_o x_o(1 - x_o/x_1)$ is the travel time for a pulse from $x_o$ to $x_1$ : $(\beta_o y_o)\omega$ is then naturally the (high frequency) lag across the filter. (<u>Note</u>: $y_o = \mu_o/2$ where $\mu_o$ is from Section 2). Also, as we expect from general theory, $\chi_T$ can be obtained from the Laplace Transform (set $\zeta = e^{i\omega y}$, $\xi_o = \beta_o x_o^2/x_1$, $\xi_1 = x_o\beta_o$, $s = i\omega$ in equation (2.5) of Section 2).

Now we can expect that low frequency pulses with their large wavelength, are unable to take advantage of the non-zero length $(x_1 - x_o)$ of the filter to remove the incompatibility $\beta_o(x_o/x_1)^2 \neq \beta_o$ : indeed we obtain $\lim_{\omega\to 0} \chi_T = 2r^2/(1+r^2)$, which is the transmission coefficient for a filter [where $r = (x_o/x_1)$]




The high frequency pulses, however, are transmitted through $x_1$, couple to the wave $x\, e^{i\omega(y - \beta_o x_o^2/x)}$ which grows WKB to a factor $(x_o/x_1)$ at $x_o$ where it is transmitted: formally $\lim_{\omega\to\infty} \chi_T = r$ (of course $2r^2/(1+r^2) \leq r$ with equality only for $r = 1$ i.e. $x_o = x_1$ and there is no filter).

These two results give the limits of response for the filter: in a theorem, $2r^2/(1+r^2) \leq |\chi_T| \leq r$ for all $\omega$. The increase with $\omega$ from $2r^2/(1+r^2)$ is not monotonic: there is an interference pattern superimposed on the coupling high frequency bias. This may be seen in Graph (6) where we plot

$|\chi_T| = r^2 / \left[ \left( \frac{1-\cos\alpha(r-1)}{\alpha^2} + r \right)^2 + \left(\frac{r-1}{\alpha}\right)^2 \left(1 - \frac{\sin\alpha(r-1)}{\alpha(r-1)}\right)^2 \right]^{\frac{1}{2}}$ versus $\alpha = -2\omega\beta_0 x_0$ for various values of $r = x_0/x_1$. ($\frac{\alpha}{2(2\pi)}$) is the number of $\beta_0$-wavelengths in the characteristic length $x_1$). From optical filter theory we expect the local maxima (see eg. graph (6)), $r = 8$: $\alpha \simeq 1.1, 1.90, 2.8, \ldots$ etc) to occur when the initial reflection off $x_0$ is 180$^0$ out of phase with the first reflection off $x_1$.

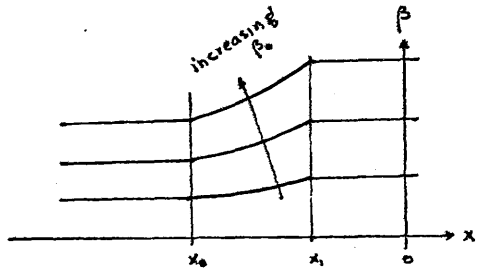
Indeed, we can show in detail that the phase difference between these reflections is given by $\psi = -\alpha(r-1) + \phi$ where $\tan\phi = \frac{\alpha(r-1)}{1+\alpha^2 r}$ and $\pi < \phi < 3\pi/2$. The product $\alpha(r-1) = 2\omega\beta_0 y_0$ is the lag due to the displacement of the coupling points $x_0$ and $x_1$ (see above) and $\phi$ is an additional phase due to incompatibilities in the media across the boundaries at $x_0$ and $x_1$ : as $\alpha \propto \omega$, $\phi$ must become constant for large $\alpha$. For $r = 8$, we obtain

$$\alpha = 1, \quad \psi = -3.14 \quad \simeq (2.(-1) + 1)\pi$$
$$\alpha = 1.87, \quad \psi = -9.47 \quad \simeq (2.(-2) + 1)\pi$$
$$\alpha = 2.75, \quad \psi = -15.77 \quad \simeq (2.(-3) + 1)\pi$$

which agree substantially with the values $\alpha = 1.1, 1.9, 2.8$ from the graph. For large $\alpha$, when $\phi \simeq$ constant $= \pi$, the local maxima are given approximately by $\alpha(r-1) = 2n\pi$, $n$ an integer. Clearly, then, the variation in $|\chi_T|$ given by $(1-\cos\alpha(r-1))/\alpha^2$ is due to the path length between $x_1$ and $x_0$, while that due to $\left[1 - \frac{\sin\alpha(r-1)}{\alpha(r-1)}\right]^2 \left(\frac{r-1}{\alpha}\right)^2$, which dies out with $\alpha (\propto \omega)$ large, is essentially a coupling phenomenon. For interference to be significant, we must be able to fit at least one $\beta_0$-wavelength into the filter $x_0 \to x_1$: when $r$ is small, it can happen that these small wavelengths are available only at frequencies high enough to overcome the coupling difficulties. Then the interference will have little amplitude: this effect is seen for $r = 2$, $r = 4$ on the graph (6).

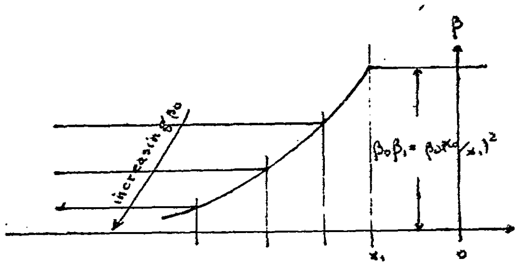
Graph (6) can be used to initiate a plausible parametric analysis of $\chi_T$. Where results have been rigorously proved, we will indicate it. Now amongst the four parameters $x_0, x_1, \beta_0, \omega$ only the combinations $\alpha = -2\omega\beta_0 x_0$ and $r = x_0/x_1$ are physically significant. Immediately we see that for $\omega$ and $\beta_0$ only their product, the wave number, is important. Thus it is sufficient to regard $\omega$ as constant:

Fix $x_0, x_1$ and vary $\beta_0$




Then $r$ is constant and $\alpha$ is proportional to $\beta_0$: the situation is identical to varying the frequency into a prescribed filter and the curves on graph (6) can be used directly. Now we can show that the filter regarded as an Ionosphere (ie. driven from the left), has a transmission coefficient porportional to $\chi_T$ : in fact $\chi_T^{Ion} = \chi_T/r^2$. Thus an Ionosphere at fixed temperature, and hence of approximately fixed extent, will shift its performance unmodified, up and down the frequency scale depending on the value of the density ie. depending on the physical density of the plasma production.

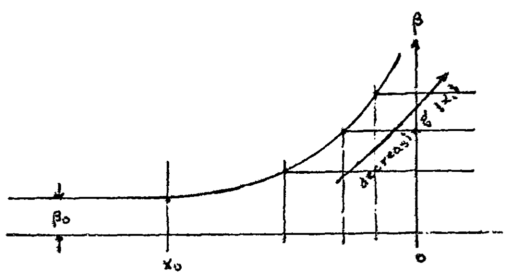
Fix $x_1$ and its density $\beta_0\beta_1 = \beta_0(\frac{x_0}{x_1})^2$ and vary $x_0$.




Thus $\alpha r = -2\omega\beta_0(\frac{x_0}{x_1})^2 x_1$ is constant. The dashed line on Graph (6) plots $\alpha r = 10$ . As $r \propto |x_0|$ , we see that $|\chi_T|$ increases with $x_0$ . Physically by increasing $x_0$ with $x_1$ and $\beta_0(x_0/x_1)^2$ fixed, we are making the output region of the filter (ie. $x \leq x_0$ ) lighter: the decreasing inertia should let through larger velocities $v$ .

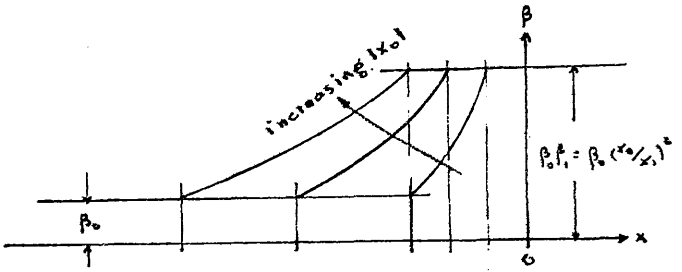
(The above variation has been rigorously proved.)

Fix $x_0$ and its density $\beta_0$ , and vary $x_1$ .



Then $\alpha$ is constant and $r \propto |x_1|^{-1}$ . From the graph we see that $|\alpha_T|$ will increase as $x_1 \to 0$ . In interpreting this result we remember that as $x_1 \to 0$ the region $x > x_1$ is becoming heavier: clearly then, a unit movement in $x > x_1$ should produce a larger response in a medium ( $< x_0$ ) that is, relatively, becoming lighter. Now if we have an Iosphere of a given temperature, so that $x_0$ is fixed, then transmission from Io will be better, the higher the density at $x = 0$ . Thus from a consideration of transmission alone, apart from considerations of generation, we see that a significant Io effect requires a substantial Iosphere.

Lastly, fix the density $\beta_0$ at $x_0$ and the density $\beta_0 (\frac{x_0}{x_1})^2$ at $x_1$ and vary $x_0$




Then $r$ is fixed and $\alpha$ varies proportional to $x_0$ : again the situation is that of varying the frequency into a given filter. Thus to vary the temperature and hence the extent of an Ionosphere of prescribed density limits $\beta_0$ and $\beta_0 (x_0/x_1)^2$, is to move its performance unmodified along the frequency scale.

The final consideration in this section is energy flux. Before proceeding we must make the energy concept precise in the magnetohydrodynamic context: using the basic equations $v_x = b_y$ and $\beta^2 v_y = b_x$ we get

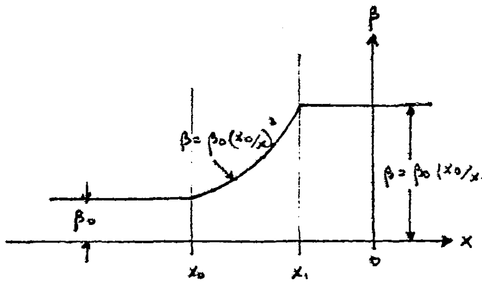
$$\left(\frac{b^2}{2} + \frac{\beta^2 v^2}{2}\right)_y = b b_y + \beta^2 v v_y = b v_x + v b_x = (bv)_x$$

ie. $$\left(\frac{b^2}{2} + \frac{\beta^2 v^2}{2}\right)_y = (bv)_x \quad \text{———} \quad (4.4)$$

Equation (4.4) is the Poynting theorem neglecting displacement currents: $b^2/2$ is the magnetostatic energy, $\beta^2 v^2/2$ is kinetic energy and $(-bv) \propto b E_z$ is the Poynting energy flux. The familiar interpretation of (4.4) is that a non-zero gradient in the flux ( $-vb$ ), which implies an unequal flow of energy into and out of an elemental length $\Delta x$ , will lead to an accumulation of energy in the length given by $\frac{\partial}{\partial y}\left(\frac{b^2}{2} + \frac{\beta^2 v^2}{2}\right)\Delta x$ . Apparently an electrostatic energy $\frac{\epsilon_0 |\vec{E}|^2}{2}$ is unimportant in magnetohydrodynamics: indeed, neglecting displacement currents, we get from Ampéres law in (1.1) (i), $0 = \nabla\cdot(\nabla \times \vec{B}) = \nabla\cdot\left[\frac{1}{c^2\epsilon_0}\vec{J}\right] = -\left[\frac{1}{c^2\epsilon_0}\right]\dot{\rho}$ , where the last equality conserves charge. Hence $\dot{r} = 0$. Thus, magnetohydrodynamic motions cannot alter an existing charge distribution: consequently the electrostatic energy $\frac{\epsilon_0|\vec{E}|^2}{2}$ cannot change in a magnetohydrodynamic process and we neglect it.

From (4.1), we see that in a wave medium ( $x<x_0$, say) a pulse $v = a_1 e^{i\omega(y-\beta_0 x)}$ travels with $b = -\beta_0 a_1 e^{i\omega(y-\beta_0 x)}$, which is the familiar result giving equipartition of energy $\frac{\beta_0^2|v|^2}{2} = \frac{|b|^2}{2}$. In the filter, however, a wave $b_1 x e^{i\omega(y+\beta_0 x_0^2/x)}$ must travel with $b = -b_1(\beta_0 x_0^2/x + i/\omega)\, e^{i\omega(y+\beta_0 x_0^2/x)}$ : as $\frac{\beta^2|v|^2}{2} = \frac{\beta_0^2}{2}(x_0^4/x^3)$ and $|b|^2/2 = \frac{\beta_0^2}{2}\frac{x_0^4}{x^2} + \frac{1}{2\omega^2}$ there can be equipartition of energy only for large $\omega$ ie. in a ray theory. (cf Bazer and Harley, Geometric Hydromagnetics (1963): JGR 68 no.1 p147-174). Indeed for small $\omega$, $|b|^2/2 >> \beta^2|v|^2/2$. (we remember from Section 3 that the filter generates wavenumber $v_x$ and hence $b_y$ : in fact $b_y = v_x = b_1 e^{i\omega(y+\beta_0 x_0^2/x)}\left(1 - \frac{i\omega\beta_0 x_0^2}{x}\right)$, where $(-i\omega\beta_0 x_0^2/x)$ is due to the curvature in the driver and the 1 is the effect of the medium. When $\omega$ is small $b = \int b_y\, dy \simeq b_1 \frac{e^{i\omega(y+\beta_0 x_0^2/x)}}{i\omega}$ can grow large in the long period $2\pi/\omega$ ). The fact that $|b| >> |v|$ for low frequencies, is important in the later theory.

It is interesting to follow $v = x e^{i\omega(y+\beta_0 x_0^2/x)}$, $b = -\left[(\beta_0 x_0^2/x) + (i/\omega)\right] e^{i\omega(y+\frac{\beta_0 x_0^2}{x})}$ into the filter




Taking real parts, we have the waves

$$v = x\cos[\omega(y+\beta_0 x_0^2/x)] \quad \& \quad b = \left[\left(\frac{\beta_0 x_0^2}{x}\right)^2 + \frac{1}{\omega^2}\right]^{1/2}\cos(\omega y + \omega\beta_0 x_0^2/x + \phi)$$

where $\tan\phi = x/\omega\beta_0 x_0^2$, $-\pi/2 < \phi < 0$. If $x_1$ is small, then as the wave approaches $x_1 \simeq 0$, $b$ tends to $\left|\frac{\beta_0 x_0^2}{x}\right|\cos\omega(y+\frac{\beta_0 x_0^2}{x})$ and the motion become WKB (near $x_1$ the wave has steepened to a wavelength small with respect to the scale of the filter and the physics follows).

The energy density is proportional to

$$\beta^2 v^2 + b^2 = \left(\theta^2 + \frac{1}{2\omega^2}\right) + \frac{\theta^2}{2}\cos(2\tau) + \left(\frac{\theta^2}{2} + \frac{1}{2\omega^2}\right)\cos 2(\tau+\phi)$$

[ where $\tau = \omega(y + \beta_0 x_0^2/x)$ is a local time for a fixed $x$ ]
$\&\ \theta = -\beta_0 x_0^2/x$

$$= \left(\theta^2 + \frac{1}{2\omega^2}\right) + \left\{\left[\frac{\theta^2}{2} + \left(\frac{\theta^2}{2} + \frac{1}{2\omega^2}\right)\cos 2\phi\right]^2 + \left(\theta^2/2 + 1/2\omega^2\right)^2 \sin^2 2\phi\right\}^{1/2} \cos(2\tau + \psi)$$

where $\cos\psi = \left[\theta^2/2 + \left(\theta^2/2 + \frac{1}{2\omega^2}\right)\cos 2\phi\right] \Big/ \left[\left(\theta^2/2 + \cos 2\phi\left(\theta^2/2 + \frac{1}{2\omega^2}\right)\right)^2 + \left(\theta^2/2 + \frac{1}{2\omega^2}\right)^2 \sin^2 2\phi\right]^{\frac{1}{2}}$

and $\sin\psi = \left[\left(\theta^2/2 + \frac{1}{2\omega^2}\right)\sin 2\phi\right] \Big/ \left[\left(\theta^2/2 + \cos 2\phi\left(\theta^2/2 + 1/2\omega^2\right)\right)^2 + \left(\theta^2/2 + \frac{1}{2\omega^2}\right)^2 \sin^2 2\phi\right]^{\frac{1}{2}}$

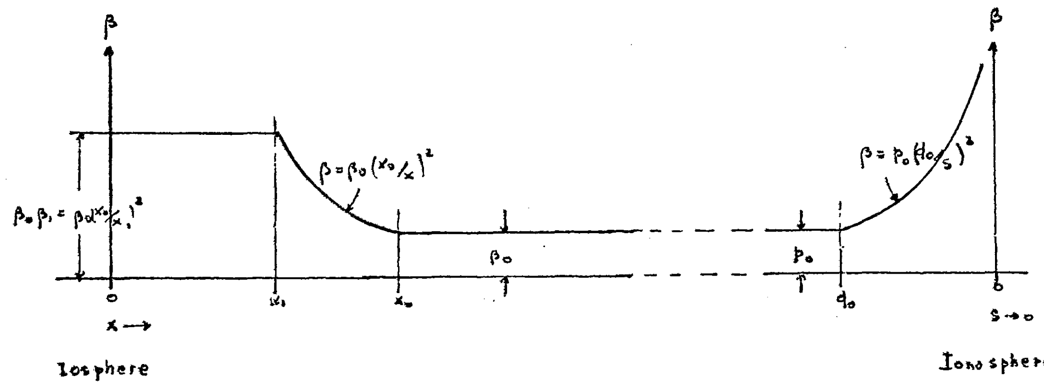
Thus at a fixed point $x$ there is in a period a maximum energy density proportional to $\left(\theta^2 + \frac{1}{2\omega^2}\right) + \left[\left(\frac{\theta^2}{2} + \left(\frac{\theta^2}{2} + \frac{1}{2\omega^2}\right)(\cos 2\phi)\right)^2 + \left(\frac{\theta^2}{2} + \frac{1}{2\omega^2}\right)^2 \sin^2 2\phi\right]^{\frac{1}{2}}$

$$= \left(\theta^2 + \frac{1}{2\omega^2}\right) + \left(\theta^4 + \frac{1}{(2\omega^2)^2}\right)^{\frac{1}{2}} .$$

This function increases as $x \to x_1$. Thus at a fixed frequency, the time averaged energy density at a point increases into the Ionosphere. Verily then the energy is localized as it slows down into the higher densities!

The energies $\frac{\beta^2 v^2}{2}$ and $\frac{b^2}{2}$ reach maxima $\left[\frac{\theta^2}{2}\right]$ and $\left[\frac{\theta^2 + \frac{1}{\omega^2}}{2}\right]$, respectively, in the period $\frac{2\pi}{\omega}$. Individually they increase into the Ionosphere owing to the localization but close to $x=0$ the ratio magnetic energy/kinetic energy $= 1 + \frac{1}{(\theta\omega)^2}$ decreases to 1. (We have shown above that near $x_1 \simeq 0$ the motion becomes WKB). An important parameter in Goertz's theory (see PhD Thesis, Rhodes University) of the decameter radiation is the magnetic energy per particle $= \left(\frac{\theta^2 + \omega^{-2}}{2}\right)/\beta^2 = x^2\left(1 + x^2/[(\beta_0\omega)^2 x_0^4]\right)$. This quantity decreases, however, as $x \to 0$ into the Ionosphere and it is not obvious that the waves from Io give rise to significant Ionospheric events. The discussion of possible Ionospheric instabilities which can generate and/or maser decameter radiation, is delicate (see Goertz, ibid).

The problems of energy transfer from Io to the Ionosphere can be gauged from the following preliminary discussion (see Section (6) for details). Consider the density along Io's flux line



(note the location of $x_1, x_0$ in the Iosphere (cf. fig (4)): $(-s)$ is the distance variable in the Ionosphere, measured from the right so that $s = q_0 < 0$ when $\beta = \rho_0$ ).

Now from (1.4) we see that time $t$ is normalized to $y = \frac{B_0}{(\mu_0 \rho_0)^{\frac{1}{2}}} t$ , where $B_0$ is a magnetic field and $\rho_0$ is a characteristic density. If we work out the details of the normalization of equations (1.3) we see that the choice of $\rho_0$ is arbitrary, but $B_0$ must be the underlying magnetic field in the direction of $x$, if $x$ is to retain its significance as a length. We find it convenient to choose $\rho_0$ the same for normalization in the Iosphere and the Ionosphere: in fact we will have $\rho_0 = \rho(x_0)$ . Then $\beta_0 = 1$ but $P_0 = \left[\frac{\rho(q_0)}{\rho_0}\right]^{\frac{1}{2}}$. It is clear then that a frequency $\omega$ in Io-time $y$ corresponds to a frequency $\omega' = \omega\,(B_0 / B_{Jup})$ in the Ionosphere ( $B_0$ is the field at Io: $B_{Jup}$ is the field in the Ionospherenear Jupiter). Let $\eta_0 = B_0/B_{Jup}$ . Then $\omega' = \omega \eta_0$ .

Now consider a pulse $v = e^{i\omega(y - \beta_0\beta_1 x)}$ (from Io) travelling to the right in $x < x_1$ : $y$ gives Io-time. Then at $x_0$ we have a transmitted wave

$$a e^{i\omega(y - \beta_0 x_0)} = \left\{ \left[ (2\omega \beta_0 x_0^2/x_1)^2 \Big/ \left[ (1 + 2i\omega \beta_0 x_0)(1 - 2i\omega\beta_0 x_0^2/x_1) - e^{2i\omega\beta_0 y_0} \right] \right] e^{i y_0 \beta_0 \omega} \right\} e^{i\omega(y - \beta_1 x_1)}$$

(use previous expression for $x_T$ but set $(-x_0) \to x_0$ and $(-x_1) \to x_1$ : then $-y_0\beta_0 = -\beta_0 x_0 (1 - \frac{x_0}{x_1})$ is the travel time).

Now we will show (see Section (6)) that a ray theory is valid in $x_0 \to q_0$ for the frequencies of interest. Let $v = a' e^{i\omega(y' - \rho_0 q_0)}$ be the wave at $s = q_0$ ( $y'$ is Ionospheric time: of course $\omega y = \omega' y'$ ).

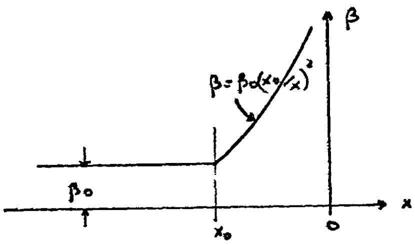
Now we have noted previously that in a ray theory there is an equipartition of energy ie. $|b|^2 = \beta^2 |v|^2$ . In fact one can show more viz. $b = -\beta v$ for a wave travelling to the right $x_0 \to q_0$ , which is the relation for a wave travelling in a constant medium $\beta$ (see equations (4.1)). Now if we remember that the time average over a period of $[(\mathrm{Re}(b))(\mathrm{Re}(v))]$ is $\frac{1}{2}\,\mathrm{Re}\,(b v^*) = -\beta |v|^2/2$ , we obtain from (4.4) the conservation of energy

$$\frac{\partial}{\partial x}\left(-\beta |v|^2/2\right) = 0$$

ie. $\beta^{-1} \beta^2 |v|^2$ is a constant. As $\beta^{-1}$ is an Alfvén velocity and $\beta^2 |v|^2$ is proportional to the kinetic energy we see that for high enough frequencies the Poynting theorem reduces to the familiar ideas (see eg. Rossi: Optics: Addison Wesley: pp456-467) of energy flow for (constant) wave media. Removing the normalization, then, high frequency energy conservation gives $V_A \rho |v_y|^2$ = constant, where $V_A$ is the local Alfvén velocity , $\rho$ the local density and $v_y$ the convected transverse velocity field (here the subscript $y$ indicates the $y$-direction, not a differentiation). Then for $x_0 \to q_0$ we have $\frac{B_0}{(\mu_0 \rho_0)^{1/2}} \rho_0 \left[\frac{B_0}{(\mu_0 \rho_0)^{\frac{1}{2}}} \cdot |a|\right]^2 = \left(\frac{B_{Jup}}{(\mu_0 \rho_0)^{\frac{1}{2}}}\right) \rho(q_0) \left(\frac{B_{Jup}}{(\mu_0 \rho_0)^{\frac{1}{2}}} \cdot |a'|\right)^2$, or $\frac{|a'|}{|a|} =$

$\eta_0^{3/2} p_0^{-1/4}$. $\therefore |a'| = |a| \eta_0^{\frac{3}{2}} p_0^{-\frac{1}{4}}$ (the factor $p_0^{-\frac{1}{4}}$ is the familiar variation (see 3rd page of Section 2) $\propto \rho^{-\frac{1}{4}}$ when the underlying field does not change: a change in the field reflects through $\eta_0^{\frac{3}{2}} = (\frac{B_0}{B_{Jup}})^{\frac{3}{2}}$).

At $q_0$, we have a transmitted wave $a'' q_0 e^{i\omega'(y' + p_0 q_0)} = a' e^{i\omega'(y' - p_0 q_0)} \left[\frac{2i\omega' p_0 q_0}{-1 + 2i\omega' p_0 q_0}\right]$ (use the result obtained previously for the Ionosphere




).

If we assemble all the factors, we obtain as a response to a pulse $v = e^{i\omega(y - \beta_0\beta_1 x)} = e^{i(\omega' y' - \omega \rho_0 \beta_1 x)}$ in the Iosphere, a wave $v = a'' s\, e^{i\omega'(y' + \beta_0 q_0^2/s)}$

where $|a''||q_0| = F_1 \cdot F_2 \cdot F_3$

and $F_1 = \left| 2i\omega' p_0 q_0 / (-1 + 2i\omega' p_0 q_0) \right|$

$F_2 = \eta_0^{3/2} p_0^{-\frac{1}{4}}$

$F_3 = \left| (2\omega \beta_0 x_0^2/x_1)^2 / [(1 + 2i\omega \beta_0 x_0)(1 - 2i\omega \beta_0 x_0^2/x_1) - e^{2i\omega\beta_0 y_0}] \right|$

Associated with the $v$ will be a $b = -a''(p_0 q_0^2/s + i/\omega')\, e^{i\omega'(y' + p_0 q_0^2/s)}$.

The average energy flow into the Ionosphere is then given by $\frac{1}{2} \mathrm{Re}(vb^*)$ $= p_0 |a'' q_0|^2/2$ which is equal to the kinetic energy at $q_0$.

The following limits are available: $\lim_{\omega \to 0} |v(q_0)| = \lim_{\omega \to 0} |a'' q_0| = 0$, $\lim_{\omega \to \infty} |v(q_0)| = \eta_0^{\frac{3}{2}} p_0^{-\frac{1}{4}} (\frac{x_0}{x_1})$

and $\lim_{\omega \to 0} |b(q_0)| = [2p_0] \times [\eta_0^{3/2} p_0^{-\frac{1}{4}}] \times [2/1 + (x_1/x_0)^2]$

$$= (\eta_0 p_0^{\frac{1}{2}})^{3/2} \frac{4}{1 + (x_1/x_0)^2},$$

$$\lim_{\omega \to \infty} |b(q_0)| = (p_0 |q_0|) \times \frac{1}{|q_0|} \times (\eta_0^{\frac{3}{2}} p_0^{-\frac{1}{4}}) \times (\frac{x_0}{x_1}) = (\eta_0 p_0^{\frac{1}{2}})^{\frac{3}{2}} (\frac{x_0}{x_1})$$

We see immediately that for energy transfer there is a total system bias (except for an interference effect in $F_3$ : see $|X_T|$) towards high frequency. This is not entirely obvious because if $4/(1 + (x_1/x_0)^2) > (\frac{x_0}{x_1})$ ie $x_0/x_1 <$ $2 + \sqrt{3} = 3.73$, $|b(q_0)|$ has a low frequency bias (we remember that low frequencies can generate large magnetic fields in the Ionosphere): as $|v|$ has a high frequency bias it is a matter of detail whether the flux $\frac{1}{2} \mathrm{Re}(vb^*)$

is favoured by low or high frequency. We get, above, $\frac{1}{2}\,\mathrm{Re}\,(vb^{*}) = \frac{p_0}{2}\,|v(q_0)|^2$: apparently the magnetic field energy due to the wavenumber effect of the Ionosphere(ie. the $\left(\frac{i}{\omega'}\right)$ in $\left|\frac{p_0 q_0^2}{s} + \frac{i}{\omega'}\right|$ ) is not transportable. The energy due to $|a''|^2 \left.\frac{|p_0 q_0^2/s|^2}{2}\right|_{s=q_0} = |a''|^2 p_0^2 q_0^2/2$, however, is portable as we see in $1/2\,\mathrm{Re}\,(vb^{*}) = \frac{1}{2}\left[p_0|v(q_0)|^2\right] = \frac{1}{2}\left[p_0\frac{|a''q_0|^2}{2} + p_0\frac{|a''q_0|^2}{2}\right]$(we remember that the factor $1/2$ in $\frac{1}{2}\,\mathrm{Re}\,(vb^{*})$ is a number obtained in calculating the time average): in fact we see that the generalization of the equipartition of energy in a wave medium, becomes the equipartition of portable energy in the medium $p_0\left(\frac{q_0}{s}\right)^2$.

All these considerations, however, are naïve: the higher frequencies are not guided efficiently along the field lines (see Section (6)): this important effect should be incorporated in $F_2$ .

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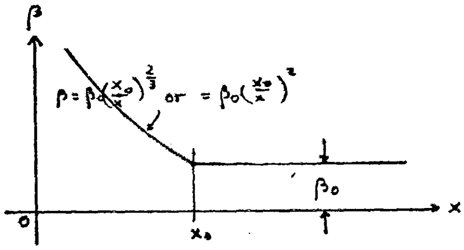
SECTION 5.

## THE DENSITY LAWS $\beta^2 = \beta_0^2 \left(\frac{x_0}{x}\right)^{\frac{4}{3}}$ AND $\beta^2 = \beta_0^2 \left(\frac{x_0}{x}\right)^2$

In Section 2 we singled out the law $\beta^2 = \left(\frac{x_0}{x}\right)^4$ as an analytically convenient density variation with an infinite travel time $x_0 \to x = 0$ ie. in the notation of Section 2, $\lim_{x\to 0} T_\delta(x) = \infty$. In this section we consider $\beta = (x_0/x)^{2/3}$ which has a finite travel time $\lim_{x\to 0} T_\delta(x) = x_0/1-\frac{2}{3} = 3x_0$ (see Section 2) and the singular density $\beta = (x_0/x)$. We remember that $\beta = (x_0/x)$ is the dividing line between densities with finite travel times and those with infinite travel times (in the model $\delta < 1$ and $\delta > 1$ respectively). $\beta = (x_0/x)$ itself has an infinite travel time $\lim_{x\to 0} T_\delta(x) = \lim_{x\to 0} \int_{x_0}^{x} (-\beta)\,dx = \lim_{x\to 0} x_0 \ln (x_0/x) = \infty$. The analysis will not be in the same detail as for the inverse fourth power: in particular we only consider an Ionosphere without reflections ie.




where we allow for $\beta_0$ not necessarily = 1.

Now from Section 2 the law $\beta = \beta_0\left(\frac{x_0}{x}\right)^{\frac{2}{3}}$ is obtained by setting $m = -2$ in the exponent $\delta = \frac{m}{m-1} = \frac{2}{3}$. There we show that the general solution to the GAE $v_{xx} - \beta_0^2 (x_0/x)^{4/3} v_{yy} = v_{xx} - \left(\frac{\beta_0^{3/2} x_0}{x}\right)^{\frac{4}{3}} v_{yy} = 0$ is a primitive $v = \int \omega \, d(\xi)^2$ of $\omega$ where $\omega$ is the general solution of the wave equation $\omega_{\xi\xi} - \omega_{yy} = 0$ and $\xi = \pm 3 (\beta_0^{3/2} x_0)^{2/3} x^{1/3} = \pm 3\beta_0 x_0 \left(\frac{x}{x_0}\right)^{1/3}$. Remembering the $\omega_{\xi\xi}$ is a solution of the wave equation whenever $\omega$ is, we can by integrating by parts, write the general solution $v = \xi \omega_\xi - \omega$. Hence, choosing $\xi = 3\beta_0 x_0 \left(\frac{x}{x_0}\right)^{\frac{1}{3}}$ we can write the general solution of the GAE as

$$v = \left[ 3\beta_0 x_0 \left(\frac{x}{x_0}\right)^{\frac{1}{3}} f'\left(3\beta_0 x_0 \left(\frac{x}{x_0}\right)^{\frac{1}{3}} - y\right) - f\left(3\beta_0 x_0 \left(\frac{x}{x_0}\right)^{\frac{1}{3}} - y\right) \right] + \left[ 3\beta_0 x_0 (x/x_0)^{1/3} g'\left(3\beta_0 x_0 (x/x_0)^{1/3} + y\right) - g\left(3\beta_0 x_0 (x/x_0)^{1/3} + y\right) \right]$$

where $f$ and $g$ are arbitrary except for some obvious mathematical properties. The motion $v = 3\beta_0 x_0 \left(\frac{x}{x_0}\right)^{\frac{1}{3}} g'\left(3\beta_0 x_0 \left(\frac{x}{x_0}\right)^{\frac{1}{3}} + y\right) - g\left(3\beta_0 x_0 \left(\frac{x}{x_0}\right)^{\frac{1}{3}} + y\right)$

$= \alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} g' [\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} + y] - g[\alpha_0 (x/x_0)^{\frac{1}{3}} + y]$

(where $\alpha_0 = 3\beta_0 x_0$ ), is from right to left. $\alpha_0$ is the travel time from $x_0$ to $0$ (we have obtained this result above for the case $\beta_0 = 1$ ie. $\lim_{x\to 0} T_{\frac{1}{3}}(x) = 3x_0$). If there is much curvature in the pulse (ie. a high frequency motion) then the term $\alpha_0 (x/x_0)^{1/3} g'(\alpha (\frac{x}{x_0})^{\frac{1}{3}} + y)$ predominates in $v$. But $\varphi(x) \propto \beta^4 \propto x^{-4/3}$ ie $\varphi^{-1/4} \propto x^{\frac{1}{3}}$ : thus we have again, as we expect, a ray theory at the high frequencies (see Section 2, third page). A large pulse, at low frequency, however, travels through the medium, moving always at the local velocity $\beta^{-1} = \beta_0^{-1} (\frac{x_0}{x})^{-2/3}$ , without change of shape ie. $v = -g[\alpha_0 (x/x_0)^{1/3} + y]$
$= -g[\int \beta dx + y]$.
The pulse $\alpha_0 (x/x_0)^{1/3} f'(\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} - y) - f(\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} - y)$ moves in a similar fashion to the right.

Let us suppose there is a driver $f_0(y + \beta_0 x)$ moving in from the right in $x > x_0$, and reaching $x_0$ at $y = 0$. Let $\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} g'(\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} + y) - g(\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} + y)$ be transmitted into the Ionosphere while $f_1(y - \beta_0 x)$ is reflected back into $x > x_0$. At $x_0$, the usual boundary conditions, yield

$$\left.\begin{aligned} \alpha_0 g'(\alpha_0 + y) - g(\alpha_0 + y) &= f_0(y + \beta_0 x_0) + f_1(y - \beta_0 x_0) \\ \frac{\alpha_0^2}{3x_0} g''(\alpha_0 + y) &= \beta_0 f_0'(y + \beta_0 x_0) - \beta_0 f_1'(y - \beta_1 x_0) \end{aligned}\right] \quad (5.1)$$

Solving, we get a first integral $g'(\alpha_0 + y) = g'(\alpha_0) e^{y/2\alpha_0} + \frac{e^{y/2\alpha_0}}{\alpha_0} \int_0^y e^{-y/2\alpha_0} \frac{d}{dy} f_0(y + \beta_0 x_0)\, dy.$ (5.2)

As $\frac{1}{2\alpha_0} > 0$, $e^{y/2\alpha_0}$ increases with time. This is reminiscent of the behaviour of the filter $\beta^2 = (x_0/x)^4$ in Section 2, when driven from the high density region. There the travel time to infinity is ~~in~~finite and it is the reflections, returning always in finite time, that give the convergence. But here the travel time to $x = 0$ is finite so that, in this case it is the reflections off $x = 0$ that give the stability.

Equation (5.2) should be compared with (3.2).

The second integral is

$$g(\alpha_0 + y) = g(\alpha_0) + 2\alpha_0 g'(\alpha_0)(e^{y/2\alpha_0} - 1) + \frac{1}{\alpha_0} \int_0^y e^{y/2\alpha_0} dy' \int_0^{y'} e^{-y''/2\alpha_0} \frac{d}{dy''} f_0(y'' + \beta_0 x_0)\, dy'' \quad (5.3)$$

Now we assume that before incidence there is no field in the medium ie $\alpha_0 (x/x_0)^{1/3} g'[\alpha_0 (x/x_0)^{\frac{1}{3}}] = g[\alpha_0 (x/x_0)^{1/3}]$

$\therefore\ g = A[\alpha_0 (x/x_0)^{\frac{1}{3}}]$ where $A$ is some constant.

Now let us choose a point $0 < x^* < x_0$. Then, as the wave travels at a finite velocity, the there exists a small time $y^* > 0$ such that

$$\alpha_0 (x^*/x_0)^{\frac{1}{3}} g'(\alpha_0 (x^*/x_0)^{\frac{1}{3}} + y^*) - g(\alpha_0 (x^*/x_0)^{\frac{1}{3}} + y^*) = 0.$$

Now where $\zeta$ is a number in the range $[0, \alpha_0]$, we have shown above that $g(\zeta) = A(\zeta)$. Also as $x^* < x_0$, $y^*$ can be supposed chosen small enough that $\alpha_0 (\frac{x^*}{x_0})^{1/3} + y^* < \alpha_0$ $\therefore g[\alpha_0 (\frac{x^*}{x_0})^{\frac{1}{3}} + y^*] = A(\alpha_0 (\frac{x^*}{x_0})^{\frac{1}{3}} + y^*)$ & $g'(\alpha_0 (\frac{x^*}{x_0})^{\frac{1}{3}} + y^*) = A$

But then $\alpha_0 (\frac{x^*}{x_0})^{\frac{1}{3}} g'(\alpha_0 (\frac{x^*}{x_0})^{\frac{1}{3}} + y^*) - g(\alpha_0 (x^*/x_0)^{\frac{1}{3}} + y^*)$ $= \alpha_0 (\frac{x^*}{x_0})^{1/3} A - A(\alpha_0 (\frac{x^*}{x_0})^{\frac{1}{3}} + y^*) = -Ay^*$, which equals zero only if $A = 0$ (we have specifically chosen $y^* > 0$).

$\therefore g(\zeta) = g'(\zeta) = 0$ for $0 \le \zeta \le \alpha_0$ $\therefore$ in particular $g(\alpha_0) = g'(\alpha_0) = 0$.

$\therefore$ (5.2) and (5.4) become

$$\left.\begin{aligned} g'(\alpha_0 + y) &= \frac{e^{y/2\alpha_0}}{\alpha_0} \int_0^y e^{-y/2\alpha_0} \frac{d}{dy} f_0(y + \beta_0 x_0)\, dy \\ \text{\&} \quad g(\alpha_0 + y) &= \frac{1}{\alpha_0} \int_0^y e^{y'/2\alpha_0} dy' \int_0^{y'} e^{-y''/2\alpha_0} \frac{d}{dy''} f_0(y'' + \beta_0 x_0)\, dy''. \end{aligned}\right\} \text{(5.4)}$$

We remember that these expressions are true only for $0 \le y \le 2\alpha_0$: at $2\alpha_0$ the reflection off $x = 0$ arrives at $x_0$. In addition, we remark that these expressions are also valid in the generalized function sense, though this is not obvious from the above derivation.

In particular, let us look at a shock $f_0(y + \beta_0 x) = \mathcal{U}(y + \beta_0(x - x_0))$, $\mathcal{U}$ the unit Heaviside, arriving at $x_0$ at time $y = 0$.

Then $f_0(y + \beta_0 x_0) = \mathcal{U}(y)$ and $\frac{d}{dy} f_0(y + \beta_0 x_0)$ is the Dirac delta $\delta(y)$.
$\therefore$ from (5.4) we get $\alpha_0 g'(\alpha_0 + y) = e^{y/2\alpha_0}$ and $g(\alpha_0 + y) = 2(e^{y/2\alpha_0} - 1)$.
There is then a wave $v = \mathcal{U}[y - \alpha_0(1 - (\frac{x}{x_0})^{\frac{1}{3}})][\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} g'(\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} + y) - g(\alpha_0 (\frac{x}{x_0})^{\frac{1}{3}} + y)]$
$= \mathcal{U}(y - y_x)[(\frac{x}{x_0})^{\frac{1}{3}} e^{(y - y_x)/2\alpha_0} - 2(e^{(y - y_x)/2\alpha_0} - 1)]$ where
$y_x = \alpha_0(1 - (\frac{x}{x_0})^{\frac{1}{3}})$ is the time for a wave to reach a point $x < x_0$.

In the front $y = y_x$, we have $v(x, y_x) = (\frac{x}{x_0})^{\frac{1}{3}} \underset{x}{\to} 0$ so that the shock structure in $v$ is destroyed by the time the wave reaches $x = 0$.

From $v_x = b_y$ we obtain $b = \mathcal{U}(y - y_x) \beta_0 (\frac{x}{x_0})^{-\frac{1}{3}} e^{(y - y_x)/2\alpha_0}$ and in the wave front $b(x, y_x) = \beta_0 (x/x_0)^{-1/3}$. Thus as $x \to 0$, $b(x, y_x) \to \infty$: this type of behaviour was also noted for $\beta^2 = (x_0/x)^4$ in Section 3. There we suggested that shocks might give rise to Ionospheric events. Here there is an extra feature: the travel time $x_0 \to 0$ is finite $(= \alpha_0)$ whereas for $(\frac{x_0}{x})^4$ it is infinite. This makes shock-generated Ionospheric events that much more likely.

On the other hand the growth due to $e^{(y-y_x)/2\alpha_0}$ is unlikely to be physically significant: at most, $e^{(y-y_x)/2\alpha_0}$ can grow to an order of $e^{2\alpha_0/2\alpha_0} = e$ before the reflected wave begins to cancel it out.

For a sinusoid driver $f_0(y+\beta_0 x) = \mathcal{U}(y+\beta_0(x-x_0)) \sin[y+\beta_0(x-x_0)]$ we have $\alpha_0 g'(\alpha_0+y) = \left[2\omega\alpha_0 / 1+(2\omega\alpha_0)^2\right](2\omega\alpha_0 \sin\omega y - \cos\omega y + e^{y/2\alpha_0})$ and $g(\alpha_0+y) = \left[\frac{2\omega\alpha_0}{1+(2\omega\alpha_0)^2}\right]\left(2e^{y/2\alpha_0} - 2\cos\omega y - \frac{\sin\omega y}{\omega\alpha_0}\right)$

This propagates from $x_0$ into the Ionosphere as the wave

$$v = \mathcal{U}(y-y_x)\left[\frac{2\omega\alpha_0}{1+(2\omega\alpha_0)^2}\right]\Big\{\left(\tfrac{x}{x_0}\right)^{\frac{1}{3}}\left(2\omega\alpha_0 \sin\omega(y-y_x) - \cos\omega(y-y_x) + e^{(y-y_x)/2\alpha_0}\right) - \left(2e^{(y-y_x)/2\alpha_0} - 2\cos\omega(y-y_x) - \frac{\sin\omega(y-y_x)}{\omega\alpha_0}\right)\Big\}$$

where $y_x$ is as above. For high frequency $v \simeq \mathcal{U}(y-y_x)\left(\frac{x}{x_0}\right)^{\frac{1}{3}}\left[(2\omega\alpha_0)^2 / 1+(2\omega\alpha_0)^2\right]\sin[\omega(y-y_x)]$ $\simeq \mathcal{U}(y-y_x)\left(\frac{x}{x_0}\right)^{\frac{1}{3}}\sin[\omega(y-y_x)]$ so that at $x = x_0$ we have perfect transmission. This is the familiar high frequency bias.

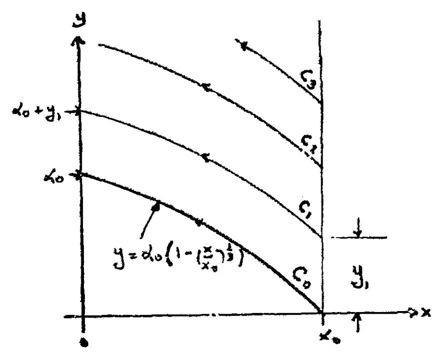
From $v_x = b_y$ we obtain

$$b = \mathcal{U}(y-y_x)\left[\frac{2\omega\alpha_0}{1+(2\omega\alpha_0)^2}\right]\left\{\beta\left(\tfrac{x_0}{x}\right)^{\frac{1}{3}}\right\}\left(2\omega\alpha_0 \sin\omega(y-y_x) - \cos\omega(y-y_x) + e^{(y-y_x)/2\alpha_0}\right)$$

The WKB growth $b \propto \varphi^{\frac{1}{4}} \propto x^{-1/3}$ is exact. We note that for $\beta = (x_0/x)^4$ (see in particular, Section 3) the motions $v$ are perfectly WKB, while those of $b$ are not: here, with $\beta = \left(\frac{x_0}{x}\right)^{\frac{2}{3}}$ the situation is reversed.

An interesting point arises as follows. We have above for $\beta = \beta_0\left(\frac{x_0}{x}\right)^{\frac{2}{3}}$, that in the front of the shock $v = \left(\frac{x}{x_0}\right)^{(\frac{1}{3})}$ and $b = \beta_0\left(\frac{x}{x_0}\right)^{-\frac{1}{3}}$ $\therefore v = \beta b$ which is the result for a wave travelling to the left in a constant medium $\beta$ (see eg. equations (4.1)). There is a similar result for $\beta = (x_0/x)^4$ : in detail, we can use the results at the end of Section 3 to show $v = -\beta b$ in the wave front (the minus sign is for a wave travelling to the right). The problem is to reconcile these results with the relationship $db = \pm\beta\, dv$ giving rise to a departition $\pm v\, d\beta$ along a characteristic, as mentioned in Section 2.

This can be done as follows. The characteristics are solutions of $dy = \pm\beta\, dx$ in the $x, y$ plane. Typically for $\beta = \beta_0\left(\frac{x_0}{x}\right)^{\frac{2}{3}}$ we can draw characteristics $dy = -\beta_0\left(\frac{x_0}{x}\right)^{2/3} dx$



An event at $x=x_0$ at time $y=0$ travels along the characteristic $c_0$ and reaches $x_0$ at $y=\alpha_0$ [which is the travel time $\lim_{x\to 0} \tau_\xi(x)$: see Section 2]: an event at $x=x_0$ at time $y=y_1$ travels along the characteristic $c_1$ and reaches $x=0$ at time $\alpha_0+y_1$ etc. Along each characteristic $db = \beta dv$. Now suppose there is a characteristic time $\frac{2\pi}{\omega}$ at $x_0$ : then in the time $\frac{2\pi}{\omega}$ the wave will have progressed a distance $dx = -\beta^{-1}dy = -\beta^{-1}(\frac{2\pi}{\omega}) = \frac{-2\pi}{\beta\omega}$ from $x_0$ . If $\omega$ is large then the distance $dx$ is small: then we can regard $\beta$ as a constant and integrate $db = \beta dv$ to get $\Delta b = \beta \Delta v$ over $dx = \Delta x$ and eventually $b=\beta v$ along the characteristic. If $\omega$ is small, however $dx$ is large on the scale of the medium and $b = \int \beta dv$ is not closely approximated by $b = \beta v$ . The point is then the following: as the shock impinges on $x_0$ , the high frequency components in its front activate $x_0$: thus along the initial characteristic $c_0$ , we have only high frequency signals and $b = \beta v$ . Along later characteristics, however, say $c_1$ or $c_2$ , the signals at are of lower frequency and hence $b = \int \beta dv$ cannot be adequately approximated by $b = \beta v$ ie. there is departition at lower frequencies. (This can be seen in the above results $v = \mathcal{U}(y-y_x)\left[ (x/x_0)^{1/3} e^{(y-y_x)/2\alpha_0} - 2\left( e^{(y-y_x)/2\alpha_0} - 1\right)\right]$, $b = \mathcal{U}(y-y_x)\, \beta_0 (\frac{x}{x_0})^{-\frac{1}{3}} e^{(y-y_x)/2\alpha_0}$ for a shock, when $y-y_x > 0$ ie. out of the front). Clearly, and as we certainly require, the method of characteristics gives a ray theory for high frequencies.

Finally we remark that in the steady state, the reflections off $x=0$ make the wave stand in the Ionosphere and no net energy from the driver passes beyond $x_0$ . This is the general result of Section 2 for density laws with a finite travel time $\lim_{x\to 0} \tau_\xi(x)$ . These considerations are important in Goertz's (see PhD Thesis, Rhodes University) theory for the decameter radiation.

Now let us consider the singular law $\beta = \beta_0 (\frac{x_0}{x})$ , which has infinite travel times in both directions (ie. $\lim_{x\to 0} \tau_\xi(x) = \lim_{x\to \infty} \theta_\xi(x) = \infty$ : see Section 2.).

A transformation

$$\left.\begin{aligned} u &= \ln x \\ \theta &= y/\beta_0 x_0 \end{aligned}\right\} \quad (5.5)$$

linearizes the GAE $v_{xx} - \beta^2 v_{yy} = 0$ to

$$v_{uu} - v_{\theta\theta} = v_u \quad (5.6)$$

(5.6) has elementary solutions

$$v = x^{\frac{1}{2}} e^{i(\omega\theta \pm \ln x\,[\,[(2\omega)^2-1]^{\frac{1}{2}}/2\,])} \quad (5.7)$$

when $\omega \geq \frac{1}{2}$

$$v = e^{i\omega\theta}\, x^{(1 \pm [1-(2\omega)^2]^{1/2})/2} \quad (5.8)$$

when $\omega \leq \frac{1}{2}$

(5.7) gives travelling waves: phase motions are obtained from
$\omega\theta \pm \ln x\ \left(((2\omega)^2-1)^{\frac{1}{2}}/2\right)$ (+ sign to the left: - sign to the right).
Hence it takes infinite time for the phase at $x_0$, say, to move to $x=0$ or $x=\infty$. The phase velocity is given by $v_{ph} = \left(\frac{dx}{dy}\right)_{ph} = \pm \frac{2\omega\,\beta_0^{-1}(\frac{x}{x_0})}{[(2\omega)^2-1]^{\frac{1}{2}}} = \pm\frac{2\omega\,\beta^{-1}}{[(2\omega)^2-1]^{\frac{1}{2}}}$
and the group velocity is

$$v_g = \left(\frac{dx}{dy}\right)_g = \pm\frac{d\omega}{d([(2\omega)^2-1]^{\frac{1}{2}}/2)}\,\beta^{-1} = \pm\frac{[(2\omega)^2-1]^{\frac{1}{2}}}{2\omega}\,\beta^{-1}.$$

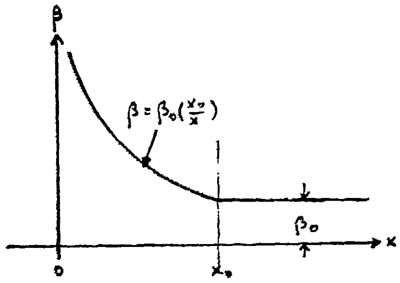
$\beta = \beta_0(x_0/x)$ is a medium with
$(v_{ph}\,v_g)^{\frac{1}{2}} = \beta^{-1}$ = characteristic speed.

We have the following limits: $\lim\limits_{\omega\to\frac{1}{2}+} v_{ph} = \infty \qquad \lim\limits_{\omega\to\infty} v_{ph} = \beta^{-1}$

$$\lim_{\omega\to\frac{1}{2}+} v_g = 0 \qquad \lim_{\omega\to\infty} v_g = \beta^{-1}$$

which indicate that the velocity of energy transfer tends to zero as $\omega\to\frac{1}{2}+$ while the higher frequencies can signal at speeds $\simeq \beta^{-1}$, the Alfvén velocity (see remarks at beginning of Section 2 on $\beta^{-1}$ group mobility).

When $\omega \le \frac{1}{2}$, the elementary solutions (5.8) give standing waves, which taken individually, cannot transport energy. We will show that an harmonic driver outside an Ionosphere $\beta_0(x_0/x)$



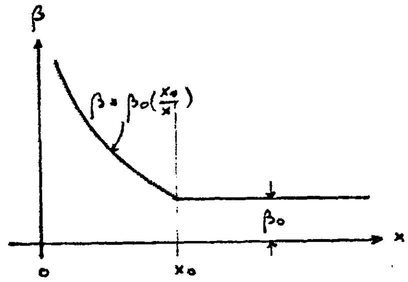

does in fact excite only one of these motions (5.8) in the Ionosphere in a steady state, so that for $\omega\le\frac{1}{2}$ it is impossible to feed energy continuously into $x<x_0$. Taken together with the $\lim\limits_{\omega\to\frac{1}{2}+} v_g = 0$ above, we say that the Ionosphere $\beta_0(\frac{x_0}{x})$ has a non-zero cutoff $\omega=\frac{1}{2}$ for energy transfer.

Thus the singular density $\beta = \beta_0(\frac{x_0}{x})$ gives a hybrid harmonic performance: for $\omega>\frac{1}{2}$ it behaves like a medium $\beta_0(\frac{x_0}{x})^{\delta}$ with $\delta>1$ : for $\omega\le\frac{1}{2}$ it behaves like a medium with $\delta<1$.

The critical frequency $\omega=\frac{1}{2}$ has the following significance. The frequency $\omega$ in $\theta$-time corresponds to a frequency $\omega_0 = \frac{\omega}{\beta_0 x_0}$ in $y$-time. Then $\omega=\frac{1}{2}$

gives $\frac{1}{\omega_0} = 2\beta_0 x_0$ . Now $2(\beta_0 x_0)$ is the time for the motion $x_0 \to 0 \to x_0$ at the Alfvén speed $\beta_0^{-1}$ . On the other hand $\frac{1}{\omega_0}$ is a characteristic time in the driver. When these two times are equal ie. $\frac{1}{\omega_0} = 2\beta_0 x_0$ , we get critical behaviour. Alternatively $\frac{\beta_0^{-1}}{2\omega_0}$ , as a speed over a frequency, gives a characteristic length in the pulse: critical behaviour occurs when this length is equal to the length of the Ionosphere ie. $\beta_0^{-1}/_{2\omega} = x_0$, equivalent to $\frac{1}{\omega_0} = 2\beta_0 x_0$.

Consider the Ionosphere




when $\omega > \frac{1}{2}$

For $x \gg x_0$ we have
$$v = a_1 e^{i\frac{\omega}{\beta_0 x_0}(y-\beta_0 x)} + a_2 e^{i\frac{\omega}{\beta_0 x_0}(y+\beta_0 x)}$$
$$= a_1 e^{i\omega(\theta - x/x_0)} + a_2 e^{i\omega(\theta + x/x_0)}$$

where we are using frequencies $\omega$ in $\theta$-time.

For $0 < x \leq x_0$
$$v = b_1 x^{\frac{1}{2}} e^{i(\omega\theta - k\ln x)} + b_2 x^{\frac{1}{2}} e^{i(\omega\theta + k\ln x)} \qquad (5.9)$$
where $k = [(2\omega)^2 - 1]^{\frac{1}{2}}/2$ is the (generalized) wave number.

Continuity of $v$ and $v_x$ at $x_0$ gives (as in Section 4).

$$Aa = Bb \qquad \text{where here} \quad a = \binom{a_1}{a_2}, \ b = \binom{b_1}{b_2}$$

$$A = \begin{bmatrix} e^{-i\omega} & e^{i\omega} \\ -\frac{i\omega}{x_0}e^{-i\omega} & \frac{i\omega}{x_0}e^{i\omega} \end{bmatrix}, \qquad B = \begin{bmatrix} x_0^{\frac{1}{2}} e^{-ik\ln x_0} & x_0^{\frac{1}{2}} e^{ik\ln x_0} \\ x_0^{-\frac{1}{2}}(\frac{1}{2}-ik)e^{-ik\ln x_0} & x_0^{-\frac{1}{2}}(\frac{1}{2}+ik)e^{ik\ln x_0} \end{bmatrix}$$

$$A^{-1} = \left[\frac{x_0}{2i\omega}\right] \begin{bmatrix} \frac{i\omega}{x_0}e^{i\omega} & -e^{i\omega} \\ \frac{i\omega}{x_0}e^{-i\omega} & e^{-i\omega} \end{bmatrix}, \qquad B^{-1} = \left[\frac{1}{2ik}\right] \begin{bmatrix} x_0^{-\frac{1}{2}}(\frac{1}{2}+ik)e^{ik\ln x_0} & -x_0^{\frac{1}{2}} e^{ik\ln x_0} \\ -x_0^{-\frac{1}{2}}(\frac{1}{2}-ik)e^{-ik\ln x_0} & x_0^{\frac{1}{2}} e^{-ik\ln x_0} \end{bmatrix}$$

For a driver $(a_2 = 1)$ moving in from the right and no sources in the Ionosphere ( $b_1 = 0$ ) we must solve $\binom{a_1}{1} = A^{-1}B\binom{0}{b_2}$

$$= \left[\frac{x_0^{\frac{1}{2}}}{4i\omega}\right] \begin{bmatrix} e^{i(\omega - k\ln x_0)}(-1+2i(\omega+k)) & e^{i(\omega+k\ln x_0)}(-1+2i(\omega-k)) \\ e^{-i(\omega+k\ln x_0)}(1+2i(\omega-k)) & e^{-i(\omega-k\ln x_0)}(1+2i(\omega+k)) \end{bmatrix} \begin{bmatrix} 0 \\ b_2 \end{bmatrix}$$

$\therefore b_2 = \frac{4i\omega}{(x_0)^{\frac{1}{2}}}\left[\frac{e^{i(\omega-\kappa\ln x_0)}}{1+2i(\omega+\kappa)}\right]$, giving a wave $\left[\frac{4i\omega}{1+2i(\omega+\kappa)}\right]\left(\frac{x}{x_0}\right)^{\frac{1}{2}} e^{i\omega} e^{i(\omega\theta+\kappa\ln(\frac{x}{x_0}))}$.
As $\omega\to\infty$, $|b_2\sqrt{x_0}| \to 1$ as it should (before comparing $4i\omega/(1+2i(\omega+\kappa))$ with the transmission coefficient $\chi$ for the Ionosphere $\beta = \beta_0(x_0/x)^2$ in Section 4, we must remember to denormalize $\omega$ in $\theta$-time to $\omega_0 = \omega/(\beta_0 x_0)$ in $y$-time etc). Also $\lim_{\omega\to\infty}(\omega-\kappa) = \lim_{\omega\to\infty} \omega - [[(2\omega)^2-1]^{\frac{1}{2}}/2] = \lim_{\omega\to\infty}\omega[1-(1-\frac{1}{2(2\omega)^2}+\cdots)] = 0$ so that the diagonal terms in $A^{-1}B$ above tend to zero as $\omega\to\infty$ : in the language of Section 4, we say that the incompatibility across $x_0$ is removable (as we expect).

As in Section 4, we calculate $b$ from $b_y = v_x$ and eventually we obtain an average energy flux into the Ionosphere $= \frac{1}{2}\mathrm{Re}(vb^*)$

$$= \beta_0\left[\tfrac{1}{2} - \left(\omega - (\omega^2-\tfrac{1}{4})^{\frac{1}{2}}\right)\right] \quad (5.10)$$

As $\omega\to\frac{1}{2}$, the flux $\to 0$ (we have shown above $\lim_{\omega\to\frac{1}{2}+} v_g = 0$ ): as $\omega\to\infty$, the flux $\to \beta_0/2$
(For high frequency, we have a ray theory $b = \beta v = \beta_0 v$ at $x_0$ : for a unit driver $v$, $\frac{1}{2}\mathrm{Re}(vb^*) = \frac{\beta_0}{2}\mathrm{Re}(vv^*) = \frac{\beta_0}{2}$ which is the limit above). (5.10) is graphed in detail in Section 6.

Now suppose $\omega \le \frac{1}{2}$.
Then in $x \ge x_0$ we still have $v = a_1 e^{i\omega(\theta-\frac{x}{x_0})} + a_2 e^{i\omega(\theta+\frac{x}{x_0})}$ but now in $0 < x \le x_0$ we have $v = b_1 x^{\frac{1}{2}} e^{i\omega\theta} x^{-\kappa'} + b_2 x^{\frac{1}{2}} e^{i\omega\theta} x^{\kappa'}$ ——— (5.11)
where $\kappa' = [1-(2\omega)^2]^{\frac{1}{2}}/2$ .

As before the boundary conditions give

$$B_c b = A a$$

where $A, a, b$ are as above for $\omega > \frac{1}{2}$ and $B_c = B_{\text{critical}} = \begin{bmatrix} x_0^{\frac{1}{2}-\kappa'} & x_0^{\frac{1}{2}+\kappa'} \\ x_0^{-\frac{1}{2}-\kappa'}(\frac{1}{2}-\kappa') & x_0^{-\frac{1}{2}+\kappa'}(\frac{1}{2}+\kappa') \end{bmatrix}$

and $B_c^{-1} = \left(\frac{1}{2\kappa'}\right)\begin{bmatrix} x_0^{-\frac{1}{2}+\kappa'}(\frac{1}{2}+\kappa') & -x_0^{\frac{1}{2}+\kappa'} \\ -x_0^{-\frac{1}{2}-\kappa'}(\frac{1}{2}-\kappa') & x_0^{\frac{1}{2}-\kappa'} \end{bmatrix}$

For a driver $a_2 = 1$ in $x > x_0$ we must solve $\binom{a_1}{1} = A^{-1}B_c\binom{b_1}{b_2}$

$$= \left[\frac{x_0^{\frac{1}{2}}}{2i\omega}\right]\begin{bmatrix} e^{i\omega}x_0^{-\kappa'}((-\frac{1}{2}+\kappa')+i\omega) & e^{i\omega}x_0^{\kappa'}((-\frac{1}{2}-\kappa')+i\omega) \\ e^{-i\omega}x_0^{-\kappa'}((+\frac{1}{2}-\kappa')+i\omega) & e^{-i\omega}x_0^{\kappa'}(\frac{1}{2}+\kappa'+i\omega) \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (5.12)$$

which gives a set of 2 equations in 3 unknowns.

Now for $\omega > \frac{1}{2}$, the wave $b_1 x^{\frac{1}{2}} e^{i(\omega\theta-\kappa\ln x)}$ (see (5.9)) travels, and transports energy, to the right: but there are no sources in $x < x_0$ ! For this reason we set $b_1 = 0$. But for $\omega \le \frac{1}{2}$ , (5.11) gives two standing waves and it is not obvious which combination of $b_1$ and $b_2$ to choose. Let us follow Lighthill (Phil. Trans. Roy. Soc. London A 252 397-430 (1960)): Lighthill replaces $\omega$ by $\omega - i\varepsilon$ where $\varepsilon > 0$ ; the system behaviour for $\varepsilon = 0$ is then obtained as the limit $\varepsilon \to 0$.

For $\omega = \omega - i\varepsilon$ the two solutions in (5.11) become

$$x^{\frac{1}{2}} e^{i\omega\theta} e^{\varepsilon\theta} x^{\pm\frac{1}{2}(1-4(\omega-i\varepsilon)^2)^{1/2}}$$
$$= x^{\frac{1}{2}} e^{i\omega\theta} e^{\varepsilon\theta} x^{\pm\frac{1}{2}((1-4\omega^2+4\varepsilon^2)+8i\varepsilon)^{\frac{1}{2}}}$$

Now for $\omega < \frac{1}{2}$, $[(1-4\omega^2+4\varepsilon^2)+8i\varepsilon]^{\frac{1}{2}} \simeq (1-4\omega^2+4\varepsilon^2)^{\frac{1}{2}}\left[1+\frac{4i\varepsilon}{1-4\omega^2+4\varepsilon^2}\right]$ for $\varepsilon$ small enough (Note: we are using the branch of the radical $(1-4z^2)^{\frac{1}{2}}$ that gives positive values for $z$ real). Thus we see that $x^{-\frac{1}{2}(1-4(\omega-i\varepsilon)^2)^{\frac{1}{2}}}$ corresponds to a movement (and a transfer of energy) to the right when $\varepsilon$ is any positive value. Clearly we must set $b_1 = 0$ in (5.12).

Then $b_2 x^{\frac{1}{2}+k'} e^{i\omega\theta} = \left(\frac{x}{x_0}\right)^{\frac{1}{2}+k'} e^{i\omega\theta}\left[\frac{2i\omega e^{i\omega}}{\frac{1}{2}+k'+i\omega}\right]$
and the reflected wave $a_1 e^{i\omega(\theta-\frac{x}{x_0})} = \left[\frac{(-1/2-k')+i\omega}{(1/2+k')+i\omega}\right] e^{2i\omega} e^{i\omega(\theta-\frac{x}{x_0})}$

which has magnitude $=1$ for all $\omega$. As $b_2 x^{\frac{1}{2}+k'} e^{i\omega\theta}$ is a standing wave, we see in detail that for $\omega < \frac{1}{2}$, no net energy can be passed into the Ionosphere continuously in the steady state (we mentioned this result at the beginning of the analysis of $\rho = \rho_0\left(\frac{x_0}{x}\right)$ in this section). The energy in the driver $a_2 e^{i\omega(\theta+\frac{x}{x_0})} = e^{i\omega(\theta+\frac{x}{x_0})}$ is carried away in the reflection $a_1 e^{i\omega(\theta-x/x_0)}$ which has magnitude $\left[\frac{(-\frac{1}{2}-k')^2+\omega^2}{(\frac{1}{2}+k')^2+\omega^2}\right]^{\frac{1}{2}} = 1$ for all $\omega \le \frac{1}{2}$ (as it must!).

Finally we tabulate some results (easy to derive from the above theory):

for $\omega > \frac{1}{2}$, $|v(x_0)| = \frac{4\omega}{[1+[2(\omega+k)]^2]^{\frac{1}{2}}} = 2\omega^{\frac{1}{2}}[2\omega - [(2\omega)^2-1]^{\frac{1}{2}}]^{\frac{1}{2}}$

$\omega < \frac{1}{2}$, $|v(x_0)| = \frac{2\omega}{[(\frac{1}{2}+k')^2+\omega^2]^{\frac{1}{2}}} = \sqrt{2}\,[1-[1-(2\omega)^2]^{\frac{1}{2}}]^{\frac{1}{2}}$

The following can be proved: $\lim_{\omega\to\infty}|v(x_0)| = 1$ (as we expect), $\lim_{\omega\to\frac{1}{2}\pm}|v(x_0)| = \sqrt{2}$ $\lim_{\omega\to 0}|v(x_0)| = 0$ :

$$\frac{d}{d\omega}|v(x_0)| > 0 \quad \text{for} \quad \omega < \tfrac{1}{2} \quad ; \quad \frac{d}{d\omega}|v(x_0)| < 0 \quad \text{for} \quad \omega > \tfrac{1}{2}.$$

Thus $|v(x_0)|$ has a maximum for $\omega = \frac{1}{2}$; at that frequency, however, there is no energy flow. Over the entire spectrum $\omega = \frac{1}{2} \to \infty$, the value of $|v(x_0)|$ doesn't change more than a factor $\frac{1}{\sqrt{2}}$. The $\lim_{\omega\to 0}|v(x_0)| = 0$ is a particular case of the general theory for low frequency i.e. $b_j \simeq 0$ (see middle of second page of Section 3).

We can make the following statements about the general density variation $\beta^2$. Suppose we look for elementary solutions $v = e^{i\omega y} u(x)$ to the GAE $v_{xx} - \beta^2 v_{yy} = 0$: then $u$ must solve $u'' + (\beta\omega)^2 u = 0$.
The familiar change of variable $u = e^{i\phi}$ then gives $-(\phi')^2 + i\phi'' + (\beta\omega)^2 = 0$. ——(5.13)

If $|\phi''|$ is small, then (see eg. Mathews and Walker: Mathematical Methods of Physics, 2nd Edition (Benjamin): p27) $(\phi')^2 \simeq (\beta\omega)^2$ and we get the familiar WKB solution (see eg. Mathews and Walker, ibid.)

$$\phi' = \pm\beta\omega \quad , \quad \phi = \pm\int \beta\omega\, dx$$

The condition ($\phi''$) small i.e. $|\phi''| << (\beta\omega)^2$ is then $|\phi''| = \left|\omega \frac{d\beta}{dx}\right| << (\beta\omega)^2$
ie. $\left|\frac{1}{\omega\beta} \frac{d\beta}{dx}\right| << \beta$ ——(5.14)

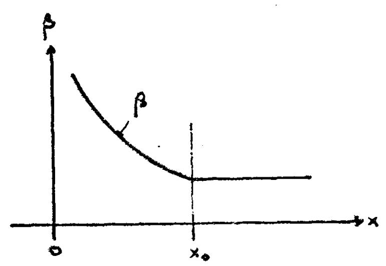
Now $\beta^{-1}$ is a speed: hence $\frac{1}{\beta}\frac{d\beta}{dx}$ is the rate of change of $\beta$ as measured by an observer moving at a speed $\beta^{-1}$ in the medium: in the characteristic time $\frac{1}{\omega}$, the observer will measure a total change $\frac{1}{\omega}\left(\frac{1}{\beta}\frac{d\beta}{dx}\right)$ : (5.14) then requires this change to be very small compared to $\beta$.

Suppose on the other hand that $|\phi'|$ is small. Then from (5.13), $\phi'' = i(\beta\omega)^2$ ie. $\phi' = i\int(\beta\omega)^2 dx$ and $\phi = i\iint(\beta\omega)^2(dx)^2$. The condition $|\phi'|$ small is then $|\phi'| = \left|\int(\beta\omega)^2 dx\right| << \beta\omega$ ie. $\left|\int \beta^2\omega\, dx\right| << \beta$ ——(5.15) (Clearly (5.15) can be obtained from (5.14) by reversing the inequality and integrating).
If we are considering solutions in a particular region $x' < x''$ then (5.15) (with the proper primitive) can be written

$$\int_{x'}^{x''} \beta^2\omega\, dx \;<<\; |\beta(x'') - \beta(x')| \quad \text{——(5.16)}$$

Now when (5.14) holds, $\phi$ is real and we get elementary solutions to the GAE which are travelling waves: under (5.16), however, $\phi$ is pure imaginary and the waves stand. We interpret this in the following way: when a pulse impinges on a region $x' \to x''$ in which the change in $\beta$ is gradual (ie. (5.14) holds), the pulse generates (WKB) travelling waves in $x' \to x''$ which can carry the energy through the medium: when the change in $\beta$ is precipitous, however, (ie. when (5.16) holds) the medium can establish standing waves in $x' \to x''$ in the characteristic time $\frac{1}{\omega}$ and all the energy is reflected back towards the driver.

Consider an Ionosphere



where $\beta(x)$ increases monotonically to $+\infty$ as $x \to 0$. Now suppose that the travel times $\lim_{x\to 0} \int_{x_0}^{x} [-\beta(x')]\,dx' = \lim_{x\to 0} \int_{x}^{x_0} \beta(x')\,dx'$

$= A$, where $A$ is a finite, positive constant.

Then where $x < x_0$, $\left[\int_x^{x_0} (\beta^2 dx')\right] \big/ \beta(x) = \int_x^{x_0} \left[\frac{\beta(x')}{\beta(x)}\right] \beta(x')\,dx' \leq \int_x^{x_0} \beta(x')\,dx' \leq A$,
for $x$ small enough ie. $\int_x^{x_0} \beta^2 \omega \, dx' \leq (\omega A)\beta$ which is $< \beta$ when $\omega < \frac{1}{A}$.

Thus when the travel time is finite, there exist (small) frequencies $\omega$ such that (5.16) is satisfied ie. such that standing waves are generated in $x < x_0$ and hence there is no net transfer of energy into the Ionosphere in the steady state.

We have shown then, that a finite travel time is sufficient to give the standing waves: it is easy to see, however, that it is not necessary. The full condition (5.16) should be used in investigating a general law $\beta$.

We can apply these ideas to the particular laws $\beta = \beta_0 \left(\frac{x_0}{x}\right)^{\delta}$. (5.14) requires $1 \ll \frac{\beta_0 x_0^{\delta}}{\delta} (x^{-\delta+1})\,\omega$ ——— (5.17).

If $\delta > 1$ ie. $-\delta+1 < 0$ then for any $\omega$ we can find an $x$ close enough to $0$ such that (5.17) holds true: moreover the closer $x$ is to zero, the more the inequality is emphasized. The WKB motions become more exact deeper into the Ionosphere (we have noticed this behaviour previously - see Section 4) and any energy which can get beyond $x = x_0$ into a region (5.17), will be carried on towards $x = 0$ without reflection. Resistance to energy transfer, if any, occurs near $x = x_0$. We notice that the greater $\delta$, ie. the steeper the medium, the higher the frequency needed to get energy past a particular point $x$ : again we have the high frequency bias!

On the other hand if $\delta < 1$ ie. $-\delta+1 > 0$ then for every $\omega$ we can find an $x$ near $0$ such that (5.17) no longer holds and such that (5.16) does hold. Thus for $\delta < 1$, energy coming in from $x > x_0$, will always find in $(0, x_0)$, a reflection point for energy.

The above ideas for $\beta_0 \left(\frac{x_0}{x}\right)^{\delta}$ can be checked against the previous work for $\delta = 2$, $\delta = 1$ and $\delta = \frac{2}{3}$ ie. $\beta^2 = \beta_0^2 \left(\frac{x_0}{x}\right)^4$, $\beta^2 = \beta_0^2 \left(\frac{x_0}{x}\right)^2$ and $\beta^2 = \beta_0^2 \left(\frac{x_0}{x}\right)^{\frac{4}{3}}$. We mention, in particular, that for the singular $\beta = \beta_0 \left(\frac{x_0}{x}\right)$, (5.17) becomes $1 \ll \beta_0 x_0 \omega$, or if $\omega^0$ is a frequency in $\theta$-time, $1 \ll \omega^0$. This estimates the critical frequency obtained previously. $(\omega^0 = \frac{1}{2})$

Lastly we mention an interesting analogue of the analytic viability of the GAE $v_{xx} - \beta_0^2 (x_0/x)^{2m/m-1} v_{yy} = 0$ for $m = 0, \pm 2, \pm 4, \ldots$ : $m = \infty$ ie $n = 2$.
We showed above that elementary solutions $v = e^{i\omega y} e^{i\phi}$ exist for the GAE, where $\phi$ solves $-(\phi')^2 + i\phi'' + [\omega\beta_0 (\frac{x_0}{x})^{\frac{m}{m-1}}]^2 = 0$ ——— (5.13)

We can convert this to a Riccati differential equation by setting $\phi' = ig$.
We get $g^2 + \omega^2 \beta_0^2 (\frac{x_0}{x})^{\frac{2m}{m-1}} = g'$ — (5.18).

But Daniel Bernoulli (see Watson: Theory of Bessel Functions: Cambridge: pp 85-6) showed that the Riccati equation is solvable in terms of elementary functions for just these exponents $n = \frac{2m}{m-1}$ of $x$ ($m = 0, \pm 2, \pm 4, \ldots$ : $m = \infty$ ie $n = 2$)

Liouville subsequently showed that, excluding the trivial case $\omega\beta_0 x_0 = 0$, only these exponents $n$ give solutions in finite terms (Watson ibid. p87).

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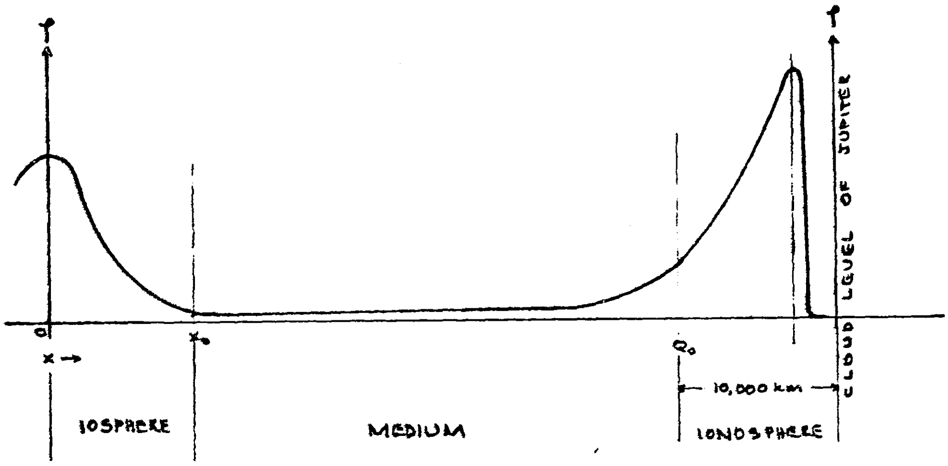
SECTION 6

CALCULATIONS

(i) GENERAL DATA

Let us consider the variation of plasma density along a typical field line through Io

fig (5.)




NOT TO SCALE

(Note: As indicated in the sketch, we refer to the region along the flux line between the Iosphere and the Ionosphere, as the Medium.)

It is understood that $x$ is set = 0 at that point along the field line where the density is a maximum in the Iosphere. Gledhill (Goddard Space Flight Centre Rept. (1967) X-615-67-296) shows that if the magnetosphere co-rotates with Jupiter, the plasma will be confined to a disk-shaped region making an angle of about $7^{\circ}$ with the rotational equatorial plane. Thus, as it orbits about Jupiter, Io will assume both positive and negative values of $x$.

In his theory of the decameter radiation Goertz (PhD Thesis: Rhodes University) calculates the distribution of plasma along a field line : he obtains (typically) the following values:-

(i) At about 1000 km above the cloud level of Jupiter the Ionosphere attains a maximum plasma density $\simeq 10^{7}$ particles /cc.

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(ii) At about 10,000 km above the cloud level the density has fallen to $\simeq 5 \times 10^{4}$/cc. We will refer to this region 1000 → 10,000 km, of length

$\simeq$ 9000 km as the "Ionosphere".

(iii) Outside the Iosphere, the density is $\simeq$ 10/cc rising to a maximum Iospheric density $\simeq$ $10^3$/cc

(iv) the flux tube is at a temperature between $1000^oK$ to $2000^oK$.

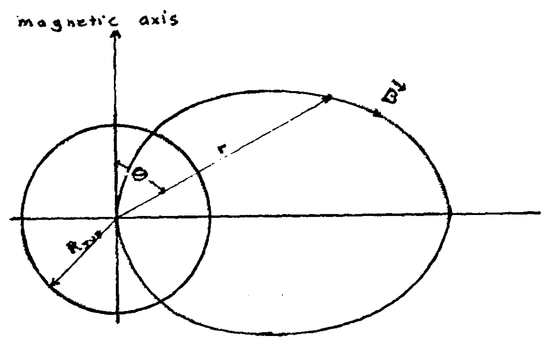
Goertz's method for calculating the Ionospheric plasma distribution involves the simultaneous solution of the heat transport equations for the electron, ion and neutral gases along with the associated momentum and chemical equations for the ion and neutral gas densities. A collisionless plasma model was adopted to calculate the density in the Medium. Inclusion of a 2-stream micro-instability in the Medium and recombination in the Iosphere leads to the formation of the plasma disk suggested by Gledhill. We will use Gledhill's equation (ibid: p16) for the distribution of plasmain the Iosphere ie. $\left[\frac{N}{N_{MAX}}\right] = \exp\left\{\frac{-2 \times (1.34 \times 10^{-6}) x^2}{T}\right\}$ ——(6.1)

when $N$ is the density (particles /cc) at $x$ (see fig. 5)

$N_{MAX}$ is maximum density (at $x=0$ ) (particles/cc)

$T$ is the absolute temperature.

Now we will assume a magnetic field $\simeq$ 10 gauss in the equatorial plane at the surface of Jupiter. This value is often assumed in Jupiter work (see eg. Carr and Gulkis, Annual Review of Astronomy and Astrophysics, Vol (8) (1970): p605). Recently Kemp et. al. (Nature 231 169 (1971)) discovered circular polarization of reflected light from Jupiter: one interpretation (Kemp et al; ibid) of this discovery, implies a magnetic field in the order of 1000 gauss or greater. If this interpretation is correct, then the entire magnetohydrodynamic analysis, as given, will need drastic revision. We will, however, assume 10 gauss. Assuming that the external Jovian field is that of a dipole and using standard results, we have an underlying field at a radius $r$ and a magnetic colatitude $\theta$ .



of $B = |\vec{B}| = B^{\circ}\left(\frac{R_J}{r}\right)^3 (3\cos^2\theta + 1)^{\frac{1}{2}}$ —(6.2)

(where here $B^{\circ} = 10$ gauss $= 10^{-3}$ Tesla from the preceeding paragraph).
($R_J = R_{Jup}$ = radius of Jupiter = 70,000 km.)

The equation for the field lines is $r = L R_J \sin^2\theta$ —(6.3) and the value of $L$ varies from line to line: Io, as mentioned in Section 1, lies on $L \simeq 6$ ie. Io orbits at about 6 x 70,000 = 420,000 km from Jupiter's centre.

The length of the field line from Jupiter's centre to Io's orbit is $\simeq$ 568,000 km (use Angerami and Thomas, JGR 69, no 21 (1964): equation (A.2))

At Io, then, we have an underlying magnetic field $\simeq B^{\circ}\left(\frac{R_J}{6R_J}\right)^3$
$= 4.63 \times 10^{-6}$ Tesla

At $x_0$, the foot of the Iosphere, the density is 10 particles/cc $\equiv 10^7/m^3 \equiv 1.68 \times 10^{-20}$ kg/m$^3$, assuming that the magnetosphere is a neutral, fully-ionized mixture of protons and electrons.

Thus the Alfvén speed $V_A$ at $x_0$ is $B_0/\sqrt{\mu_0\rho} = \frac{4.63 \times 10^{-6}}{[(4\pi \times 10^{-7}) \times 1.68 \times 10^{-20}]^{\frac{1}{2}}} = .34 \times 10^8$ m/sec $\equiv .11c$.

Along the field line and at $Q_0$ (ie. 10,000 + 70,000 = 80,000 km from the centre of the planet), the colatitude is given from $\sin^2\theta \simeq \frac{1}{6}$
ie. $\cos^2\theta \simeq 5/6$ and $\therefore B = 10^{-3} \times \left(\frac{7}{8}\right)^3 (3 \times 5/6 + 1)^{\frac{1}{2}} = 1.26 \times 10^{-3}$ Tesla.
Also the density is $5 \times 10^4$/cc $\equiv 8.38 \times 10^{-17}$ kg/m$^3$

Alfvén speed $= 1.26 \times 10^{-3} / [4\pi \times 10^{-7} \times 8.38 \times 10^{-17}]^{\frac{1}{2}} = 1.23 \times 10^8$ m/sec $\equiv .43c$

(Similar calculations for $x = 0$ and at 1000 km in the Ionosphere, give $V_A = .011c$ & $.026c$ respectively.)

Thus we see that over the entire magnetosphere, our non-relativistic treatment is likely to be a good approximation. (The important quantity is $[1-(V_A/c)]^{\frac{1}{2}}$, which, even for $V_A = .43c$, equals $.903 \simeq 1$)

In the Iosphere, the important length for disturbances is in the order of $D_{Io}$ = diameter of Io = 3000 km : Goertz (ibid.) uses 2 x $D_{Io}$ ie. 6000 km and then requires that $V_A/(2D_{Io})$ is the important frequency.

At $x = x_0$, $V_A/2D_{Io} = .34 \times 10^8/[6000 \times 10^3] \simeq 5.5$ Hz

At $x = 0$ where density = $10^3$ particles /cc, $V_A/2D_{Io} = \frac{.34 \times \left(\frac{10}{10^3}\right)^{\frac{1}{2}} \times 10^8}{6000 \times 10^3} \simeq .5$ Hz

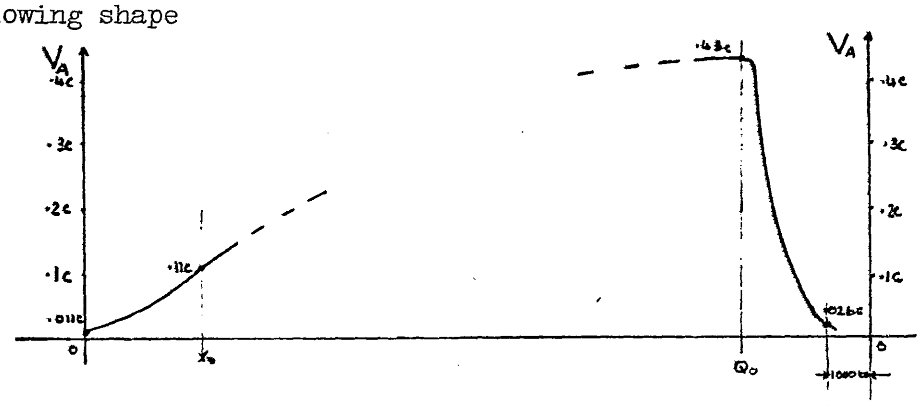
Thus we consider Io to generate in the range .5 - 5.5 Hz (hence the value 5Hz in Section 1).

At this point, we can conveniently check the applicability of the entire magnetohydrodynamic treatment. The Debye length h, is given by (see eg. Holt and Haskell: Plasma Dynamics, Macmillan: equation (9.16)

$$h = 6.9 \times 10^{-5} \times \sqrt{\frac{T}{N}} \qquad (T - {}^{o}K,\ N - cm^{-3})$$

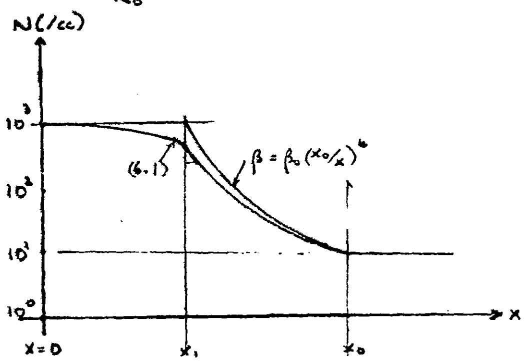
Thus at $x_o$, the point of lowest density, $h = 6.9 \times 10^{-5} \left(\frac{1500}{10}\right)^{\frac{1}{2}}$ (assuming $T = 1500^{o}K$) $= 84.5 \times 10^{-5} \cong .85 \times 10^{-3}$ km which is $<<$ 2 $D_{Io}$ = 6000 km. Also the proton gyrogrequency (at Io) $= \nu_i = \left[\frac{eB}{2\pi\, m_{proton}}\right] = \left[\frac{1.6 \times 10^{-19} \times 4.63 \times 10^{-6}}{2\pi \times 1.67 \times 10^{-27}}\right]$ $\cong 70\,Hz >> 5.5\,Hz$. Thus we have confidence in applying the magnetohydrodynamic method.

A plot of Alfvén velocity $V_A$ along the Io flux line should have the following shape




## (ii) THE IOSPHERE.

In subsection (i) we give the law $\left[\frac{N}{N_{MAX}}\right] = \exp\left\{\frac{-2.68 \times 10^{-6} \times x^2}{T}\right\}$ — (6.1), for the variation of plasma density in the Iosphere. We now show that this law can be adequately approximated by the inverse fourth power law ie. $\beta = \beta_o (x_o/x)^4$ ie. $\frac{N}{N_o} = (x_o/x)^4$ where $N_o = N(x_o)$



Now $N_{MAX} = 10^3/cc$. We will assume that the extent of the Iosphere $10^3 \rightarrow 10^1$ is determined from (6.1)

ie. $\frac{10}{10^3} = \exp\left\{-\frac{(2.68 \times 10^{-6}) \times (x_0)^2}{T}\right\}$

For T = 1000°K, this gives $x_0 \simeq$ 41,500 km

T = 2000°K $x_0 \simeq$ 58,500 km

We then calculate $x_1$ from $\frac{N_{MAX}}{N_0} = \frac{10^3}{10} = \left(\frac{x_0}{x_1}\right)^4$

to get $x_1 \simeq$ 13,100 km for T = 1000°K

$x_1 \simeq$ 18,500 km for T = 2000°K

We can get some estimate of the degree to which $\frac{N}{N_0} = \left(\frac{x_0}{x}\right)^4$ approximates (6.1) by calculating $N\Big|_{x=x_1} = N_{MAX} \exp\left\{\frac{-2.68 \times 10^{-6} \times (x_1)^2}{T}\right\}$

For T = 1000°K, we get $N\Big|_{x=x_1}$ = .632 x $10^3$ /cc

T = 2000°K $N\Big|_{x=x_1}$ = .629 x $10^3$ /cc

Thus as both these numbers are close to $10^3$/cc, $\frac{N}{N_0} = \left(\frac{x_0}{x}\right)^4$ gives a good approximation over the temperature range.

Now, as we mentioned above, Goertz's theory involves a characteristic length $\simeq$ 6000 km. The scale length for the (smaller) Ionosphere, T = 1000°K, is $\simeq$ 41,500 - 13,100 = 28,400 $\gg$ 6000 km. Thus there should be little interaction between Goertz's waves and the Ionosphere: the energy from Io will pass through the filter $x_1 \rightarrow x_0$, with only a small reduction in amplitude.

Frequencies of the order of $\frac{1}{60}$ Hz ( giving a characteristic time of 1 min., which is in the order of the time Io takes to cross its own length: see eg. Drell, Foley and Rudeman, JGR <u>70</u> (3131) (1965)), however, will have a length of $\left\{\frac{(V_A)_{x_1} + (V_A)_{x_0}}{2}\right\} / \left(\frac{1}{60}\right) = \left[\frac{.011 + .11}{2}\right] \times \frac{3 \times 10^5}{(1/60)}$

$\simeq$ 1.09 x $10^4$ km which should be well contained by the Iosphere.

We can see this in more detail from the transmission coefficient

$$|\chi_T| = r^2 \Big/ \left[\left(\frac{1-\cos\alpha(r-1)}{\alpha^2} + r\right)^2 + \left(\frac{r-1}{\alpha}\right)^2 \left(1 - \frac{\sin\alpha(r-1)}{\alpha(r-1)}\right)^2\right]^{\frac{1}{2}}$$

where $r = \frac{x_0}{x_1}$ and (here) $\alpha = (+2\omega\beta_0 x_0)$. (We remember that the calculation of $\chi_T$ was on the basis of a constant underlying magnetic field $\vec{B}_0$ : in the Iosphere, where the chief variation in $|\vec{B}_0|$ is through the angle $\theta$ in (6.2), this is a good approximation).

When $\left[\left[1-\cos\alpha(r-1)\right]/\alpha^2\right] \ll \underbrace{r}_{(6.6)}$, then $|\alpha_r| \cong r$ and there is no interaction between the wave and the filter. (6.4) can be written in the form $\left[\frac{\sin[\alpha(r-1)/2]}{[\alpha(r-1)/2]}\right]^2 \ll 2r/(r-1)^2$ . For both T = 1000°K and 2000°K we have $r = (x_0/x_1) \cong 3.16$. ∴ the above inequality becomes $\left|\frac{\sin 1.08\alpha}{1.08\alpha}\right| \ll 1.17$.

Clearly it is sufficient to consider $\frac{1}{1.08\alpha} \ll 1.17$ ie. $\alpha \gg .79$ ie. $2\omega\beta_0 x_0 \gg .79$.

Now an angular frequency $\omega$ in $y$ -time becomes (see equations (1.4)) a frequency $\nu = \frac{\omega}{2\pi}\frac{B_0}{(\mu_0\rho_0)^{\frac{1}{2}}}$ in real time $t$ .

Also $\beta_0 = \left[\frac{\rho(x_0)}{\rho_0}\right]^{\frac{1}{2}}$ $\therefore\ 2\omega\beta_0 x_0 = 2 \times \left\{(2\pi\nu)\frac{(\mu_0\rho_0)^{\frac{1}{2}}}{B_0}\right\}\left[\left(\frac{\rho(x_0)}{\rho_0}\right)^{\frac{1}{2}}\right]x_0$

$$= \frac{4\pi\nu x_0}{(V_A)_{x_0}} = \frac{4\pi\nu\ (41,500)}{.34 \times 10^5} \quad [\text{ for the Ionosphere, T = 1000°K.}]$$

$$= 15.8\,\nu \gg .79$$

ie. $\nu \gg$ 0.05 Hz (Hence the value .05 Hz in the "Introduction".)

Goertz's waves have $\nu \cong 5 \gg$ .05 Hz, but the $\frac{1}{60}$ Hz waves have $\frac{1}{60}$ =.017 < .05 Hz.

Thus the factor $F_3$ in the total transmission $|a''q_0|$ at the end of Section 4 should be $\cong$ 1.

## (iii) <u>THE MEDIUM</u>

We will show first that the GAE $v_{xx} - \beta^2 v_{yy} = 0$ (derived in Section 1 for a constant underlying field) is valid in the Medium and that a ray theory gives a good solution there.

We have obtained previously that at $x_0$ , $V_A = .34 \times 10^8$ m/sec

at $Q_0$ , $V_A = 1.23 \times 10^8$ m/sec .

(for an extreme choice of parameters Goertz calculates the point $Q_0$ with $N = 5 \times 10^4$/cc to be at 7,000 km rather than at 10,000 km above the cloud level, as we are assuming: for 7,000 km the corresponding $V_A$ would be 1.38 x $10^5$ km/sec = .46 c; it is against such an extreme case that we use .46 c in Section 1 (vide)).

Now the length of the medium = 568,000 - (length of Iosphere) - 80,000 km = 568,000 - 50,000 - 80,000 = 438,000 km (50,000 km is the average of the values $x_0$ obtained in the previous subsection for 1000°K and 2000°K).

Thus the average gradient in wavelength over the medium is $-\left[\frac{(\frac{V_A}{\nu})_{00} - (V_A/\nu)_{x_0}}{438,000}\right]$ $= -\frac{(1.23 - .34)\times 10^5}{438,000\,\nu} \cong \frac{-.20}{\nu}$ where $\nu$ is (again) the frequency in the pulse. For $\nu$ = 5 Hz, $-\frac{.20}{\nu} = -.04$. Thus in a unit length the wavelength will (on the average) hardly change.

Also, the average gradient in the underlying field is (using values from subsection (i)) $-\frac{(1.26 - .0046)\times 10^{-3}}{438,000,000} \cong -2.9\times 10^{-12}$ Tesla/m

Now the basic equations are, for polarization in the $z$ - direction, say,

$$\left.\begin{aligned} B_0\frac{\partial v_z}{\partial x} &= \frac{\partial b_z}{\partial t} \\ \frac{B_0}{\mu_0\rho}\frac{\partial b_z}{\partial x} &= \frac{\partial v_z}{\partial t}\end{aligned}\right] \text{———} (6.5)$$

(see equations (1.3)).

These equations are derived for $B_0$, the underlying field, constant. In following through the derivation preceeding (1.3) in Section (1), it is seen that (6.5) remains true when $B_0 = B_0(x)$ provided $\left|v_z\frac{dB_0}{dx}\right| << \left|B_0\frac{\partial v_z}{\partial x}\right|$ (we mentioned this result towards the end of Section (1)).

Now let $k$ be a typical wave number in the pulse.
Then $\left|v_z\frac{dB_0}{dx}\right| << \left|B_0 \partial v_z/\partial x\right|$ requires $B_0|k| \Big/ \left|\left(\frac{dB_0}{dx}\right)_{average}\right|$

$$= \left[B_0\left[\frac{2\pi\nu}{B_0/(\mu_0\rho)^{\frac{1}{2}}}\right]\right] \Big/ \left|\left(\frac{dB_0}{dx}\right)_{average}\right| = \frac{(\mu_0\rho)^{\frac{1}{2}}(2\pi\nu)}{\left|(dB_0/dx)_{average}\right|} >> 1$$

Taking $N$ an average value ie. $N \cong 10^3/cc$ ie $\rho = 1.68\times 10^{-18}$ kg/m$^3$ and using $\left|\left(\frac{dB_0}{dx}\right)_{average}\right|$ ~~average~~ $= 2.9\times 10^{-12}$ Tesla/m obtained above, we then require $3.16\,\nu >> 1$. For $\nu \cong$ 5Hz, we have 3.16 x 5 = 15.80 so that we may use (6.5) for $B_0 = B_0(x)$ with confidence.

Then eliminating, we obtain from (6.5)

$$\left(\frac{B_0^2}{\mu_0\rho}\right)\frac{\partial^2 v_z}{\partial x^2} + \left(\frac{B_0}{\mu_0\rho}\right)\left(\frac{\partial v_z}{\partial x}\right)\left(\frac{dB_0}{dx}\right) = \frac{\partial^2 v_z}{\partial t^2} \text{ ——— } (6.6)$$

The magnitude of the first term on l.h.s. of (6.6) is $\propto \frac{B_0^2}{\mu_0\rho}k^2$, where $k$ is (again) the wave number: the magnitude of the second term on l.h.s., is $\propto \frac{B_0}{\mu_0\rho}\left|\left(\frac{dB_0}{dx}\right)_{average}\right| |k|$

Their ratio is $\left(\frac{B_0^2 k^2}{\mu_0\rho}\right) \Big/ \frac{B_0}{\mu_0\rho}\left|\left(\frac{dB_0}{dx}\right)_{average}\right| |k| = \frac{B_0|k|}{\left|\left(\frac{dB_0}{dx}\right)_{average}\right|}$. But we have shown above that this ratio is small for $\nu \cong$ 5Hz. Thus we can neglect the second term on the l.h.s. of (6.6). We have then in the Medium the GAE

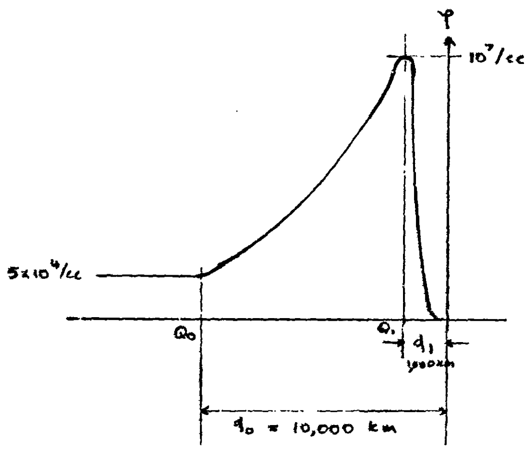
$$\frac{B_0^2}{\mu_0 \rho} \frac{\partial^2 v_z}{\partial x^2} = \frac{\partial^2 v_z}{\partial t^2}$$

Also, as we have shown above that the average gradient in wavelength is small $(\simeq 1 - .04)$ for $\nu \gtrsim$ 5Hz, we have then, finally, that a ray theory gives a good approximation in the Medium. Thus all the energy escaping from Io, will be transmitted through the Medium and impinge on the Ionosphere at $Q_0$ . The travel time $(x_0 \to Q_0)$ will be $\simeq \frac{438{,}000}{[(V_A)_{Q_0} + (V_A)_{x_0}]/2}$ $= \frac{876{,}000}{(1{\cdot}23 + {\cdot}34) \times 10^5}$ = 5.6 sec.

At this stage we must reconsider the approximations made in Section 1 of infinite conductivity and incompressibility.

Lighthill (Phil. Trans. Roy. Soc. London A <u>252</u> 397-430 (1960)) shows that both a more realistic equation for current in the plasma and the inclusion of a finite compressibility lead to a deguiding of energy along the field line. The current effect is more significant. Lighthill shows that if we incorporate a (large) Hall effect, then a disturbance in the plasma will spread out within a cone whose angle is $\arcsin(\omega/\omega_i)$ where $\omega$ is the angular frequency of the disturbance and $\omega_i$ is the ion (angular) gyrofrequency $= 2\pi\nu_i$ (see subsection (i)) at Io . For frequencies $\frac{\omega}{2\pi} \gtrsim$ 5Hz, this conical attenuation $\arcsin(\frac{5}{70}) \simeq 4^\circ$ becomes significant and the wave loses amplitude along a field line: these ideas are important in a theory of the decameter sources (see Goertz and Deift, to be published; will be referenced in Goertz, PhD Thesis, Rhodes University: see also subsection (iv)).

(iv) Lastly we consider the Ionosphere



If we assume an inverse fourth power density variation we have in the above figure $(q_1/q_0) = \left(\frac{5\times10^4}{10^7}\right)^{\frac{1}{4}} \approx .27$

As $q_0 = 10,000$ km, this gives $q_1 = 10,000 \times .27 \approx 2700$ km which is greater than 1000 km. We need a variation that is less steep. For the inverse square we get $(q_1/q_0) = (5\times10^4/10^7)^{\frac{1}{2}} \approx .07 \therefore q_1 = .07 \times 10,000 = 700$ km. The law $\beta^2 = \beta_0^2 (x_0/x)^{4/3}$, on the other hand, would give $q_1 \approx 200$ km, which is too small. We will use the inverse square law $\beta_0^2 (\frac{x_0}{x})^2$ in what follows. Also we will assume that internal reflections are not important in the Ionosphere (see Section 3): then we can extend $\beta_0^2 (x_0/x)^2$ to $x = 0-$.

Across $Q_0 \rightarrow Q_1$ the magnetic field varies as $\left(\frac{B_{Q_1}}{B_{Q_0}}\right) = \left[\frac{80,000}{71,000}\right]^3 \approx 1.44$

However $\left[\rho_{Q_1}/\rho_{Q_0}\right]^{\frac{1}{2}} = (10^7/5\times10^4)^{\frac{1}{2}} \approx 14.1$

Thus in the Ionosphere we will neglect the variation of $B_0$ with respect to that in $\rho^{\frac{1}{2}}$ in $V_A = B/(\mu_0\rho)^{\frac{1}{2}}$. Then the method of Section 5 is applicable.

The law $\beta_0^2 (x_0/x)^2$ has a cut off at a frequency $\omega_0 = \frac{1}{2}$, where $\omega_0$ is in $\theta$ -time (see Section 5).

This corresponds (see (1.4), (5.5)) to a frequency $\left[\frac{\omega_0}{2\pi} \frac{(V_A)_{Q_0}}{q_0}\right]$ $= \frac{(\frac{1}{2})}{2\pi}\left[\frac{1.13\times10^5}{10,000}\right] \approx 1$ Hz.

If we now plot the energy flux from (5.10)

$$\text{Flux} = \beta_0 \left[\tfrac{1}{2} - (\omega_0 - (\omega_0^2 - \tfrac{1}{4})^{\frac{1}{2}})\right]$$

[$\omega_0$ is an angular frequency in $\theta$ -time)

we obtain

| $\omega_0$ | Flux / $[\beta_0/2]$ |
|---|---|
| ·5 | 0 |
| ·625 | ·51 |
| ·75 | ·63 |
| 1 | ·76 |

To transmit 60%, say, of the incident radiation into the Ionosphere we need $\omega_o > .75$ ie. $\nu > 1 \times \frac{(3/4)}{(1/2)} = 1.5$Hz. Thus not all of the frequencies .5 - 5 Hz generated by Io will get into the Ionosphere.

This result, together with the deguiding of (iii) is used to give the explanation of the decameter sources mentioned in the previous subsection (see (iii) for reference).

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## APPENDIX

### CONVERGENCE OF $p(y) = \mathcal{L}^{-1}\{\bar{H}(s)\}$

It is easy to invert $\bar{H}(s)$ term by term to obtain $p(y) = \mathcal{L}^{-1}\{\bar{H}(s)\}$

$$= \sum_{k=1}^{\infty} \gamma_0^{k+1}\, \mathcal{U}(y-\mu_0 k)\, e^{-r_0(y-\mu_0 k)} \int_0^{y-\mu_0 k} \left[\frac{(y-\mu_0 k-u)^{k-1}\, u^k}{k!\,(k-1)!}\right] e^{(r_1+r_0)u}\, du$$

$$+ \gamma_0 e^{r_1 y}$$

$$- \tilde{\gamma}_0 \sum_{k=0}^{\infty} \gamma_0^{k+1}\, \mathcal{U}(y-\mu_0 k)\, e^{-r_0(y-\mu_0 k)} \int_0^{y-\mu_0 k} \left[\frac{(y-\mu_0 k-u)^{k}\, u^k}{(k!)^2}\right] e^{(r_1+r_0)u}\, du \quad \text{———} (A.1)$$

where $\mathcal{U}$ is the unit Heaviside, showing explicity that the $k^{th}$ wave reaches $x$, only after a time $\mu_0 k$. A consideration of the limit, $\lim_{y\to\infty} p(y)$ could proceed, pari passu, with a justification of the applicability of the final value theorem. Where $y_j = \mu_0 j$, we will, however, consider the reduced problem $\lim_{j\to\infty} p(y_j)$.

The sequential solution indicates a general method for series: the problems encountered, however, are essentially those of the general limit, $\lim_{y\to\infty} p(y)$. There is apparently a deep relationship between the theory of Laplace transforms and that of series.

The first term in (A.1) becomes, for $j \geq 1$, $\sum_{k=1}^{j} \gamma_0^{k+1} e^{-r_0\mu_0(j-k)} \int_0^{(j-k)\mu_0} \frac{[\mu_0(j-k)-u]^{k-1} u^k}{k!\,(k-1)!} e^{(r_1+r_0)u}\, du$

$$= \sum_{k=0}^{j-1} \gamma_0^{k+2} e^{-r_0\mu_0(j-k-1)} \int_0^{(j-k-1)\mu_0} \frac{(\mu_0(j-k-1)-u)^k u^{k+1}}{k!\,(k+1)!} e^{(r_1+r_0)u}\, du \,.$$

Now $\int_0^{(j-k-1)\mu_0} [\mu_0(j-k-1)-u]^k u^{k+1} e^{(r_1+r_0)u}\, du = \left[\frac{\mu_0(j-k-1)}{2}\right]^{2k+2} e^{\frac{(r_1+r_0)\mu_0(j-k-1)}{2}} \int_{-1}^{1} (1-w^2)^k (1+w)\, e^{-c(j-k-1)w}\, dw$

where $c = \frac{-(r_1+r_0)\mu_0}{2} = \frac{r_1^2-r_0^2}{2r_1r_0} > 0$ and we changed the variable of integration from $u$ to $v = 2u - \mu_0(j-k-1)$ to $w = v/\mu_0(j-k-1)$ when $j-k-1 \neq 0$.
Again for $(j-k-1) \neq 0$, $\int_{-1}^{1} (1-w^2)^k w\, e^{-c(j-k-1)w}\, dw$ integrates by parts to
$\frac{-c(j-k-1)}{2(k+1)} \int_{-1}^{1} (1-w^2)^{k+1} e^{-c(j-k-1)w}\, dw$

$= \left(\frac{\pi}{2c(j-k-1)}\right)^{1/2} I_{k+3/2}(c(j-k-1)) \; \frac{[-c(j-k-1)]\, k!}{[c(j-k-1)/2]^{k+1}}$ (where $I_{k+3/2}$ is a modified Bessel function - see Bell, Special Functions for Scientists and Engineers, p116 for the integral representation)
$= f_{k+1}[c(j-k-1)] \left\{[-c(j-k-1)]\, k! / [c(j-k-1)/2]^{k+1}\right\}$, where we have in effect set $\left(\frac{\pi}{2z}\right)^{1/2} I_{k+1/2}(z) = f_k(z)$. The function $f(z)$ may be termed a modified Spherical Bessel function of the first kind (see Abramowitz and Stegun: Handbook of Mathematical Functions: Dover: p443) the functions may be expressed in terms of elementary functions eg.

$f_0(z) = \sinh z/z \,, \quad f_1(z) = -\frac{\sinh z}{z^2} + \frac{\cosh z}{z} \,.$ (Abromowitz and Stegun ibid).

Similarly $\int_{-1}^{1}(1-\omega^2)^k e^{-c(j-k-1)\omega}\,d\omega = f_k(c(j-k-1)) \times \frac{2^{k+1}k!}{[c(j-k-1)]^k}$ for $(j-k-1)\neq 0$.

We may put all these results together to obtain

$$\sum_{k=0}^{j-1} \gamma_0^{k+2} e^{-r_0\mu_0(j-k-1)} \int_0^{(j-k-1)\mu_0} \frac{[\mu_0(j-k-1)-u]^k u^{k+1}}{k!\,(k+1)!} e^{(r_1+r_0)u}\,du$$

$$= \gamma_0 \sum_{k=0}^{j-1} \frac{(-a)^{k+1}}{(k+1)!} e^{b(j-k-1)} [c(j-k-1)]^{k+2} \{ f_k\, c(j-k-1) - f_{k+1}[c(j-k-1)] \}$$

where the term $j-k-1=0$ is (trivially)included. Here $0<a = \frac{-2\gamma_0}{(r_0+r_1)^2} = \frac{2r_1r_0}{(r_0+r_1)^2}$ is positive and $<1/2$, $b = \frac{(r_1-r_0)\mu_0}{2} = \frac{-(r_1-r_0)^2}{2r_1r_0} < 0$ and we are only considering $j\geq 1$. Let us denote this sum by $v_j$ for $j>0$ and define $v_0=0$.
Similarly one can evaluate the third term in (A.1) to get

$$(-r_0a_0)\sum_{k=0}^{j} \frac{(-a)^k}{k!} e^{b(j-k)} [c(j-k)]^{k+1} f_k[c(j-k)]$$

We denote this sum by $u_j$, $j\geq 0$. Here $a_0 = {2r_1r_0}/{r_0+r_1} < 0$.

The method we will use to demonstrate convergence will be to associate with the sequence $\{v_j\}$, say, a power series $\bar{v}(z) = \sum_{j=0}^{\infty} z^j v_j$.
If we form the difference series $w_j = v_{j+1}-v_j$, then $\bar{w}(z) = \sum_{j=0}^{\infty} z^j w_j$
$= (\frac{1}{z}-1)\bar{v} - \frac{1}{z}v_0$ for $z\neq 0$. But under certain conditions (with which we will concern ourselves) $\lim_{z\to 1}\bar{w}(z) = \lim_{z\to 1}\sum_{j=0}^{\infty} z^j w_j = \sum_{j=0}^{\infty} w_j =$

$$= \lim_{J\to\infty}\sum_{j=0}^{J} w_j = \lim_{J\to\infty}(v_{J+1}-v_0) = (\lim_{j\to\infty} v_j) - v_0 \quad \therefore \lim_{j\to\infty} v_j = \lim_{z\to 1}(\tfrac{1}{z}-1)\bar{v},$$

which is a final value theorem. Clearly we are working in analogy with Laplace Transform theory, a fact we could emphasize by writing $e^{-\alpha}$ for $z$ ($\alpha$ is some complex number).

Now consider a summand in $v_j$

$$\left| \sum_{k=0}^{j-1} \frac{(-a)^{k+1}}{(k+1)!} e^{b(j-k-1)} [c(j-k-1)]^{k+2} f_k[c(j-k-1)] \right|$$

$$\leq \sum_{k=0}^{j-1} \frac{a^{k+1}}{(k+1)!} e^{b(j-k-1)} [c(j-k-1)]^{k+2} f_k[c(j-k-1)] \qquad (\text{as } f_k(z)\geq 0 \text{ for } z\geq 0)$$

$$\leq f_0(c(j-1)) \sum_{k=0}^{j-1} \frac{a^{k+1}}{(k+1)!} e^{b(j-k-1)} [c(j-k-1)]^{k+2} \qquad (\text{as } f_k[c(j-k-1)] \leq f_k[c(j-1)] \leq f_0[c(j-1)]$$

$$\leq c\, f_0[c(j-1)] \sum_{k=0}^{j-1} \frac{(ac)^{k+1}}{(k+1)!} (j-k-1)^{k+2} \qquad (\text{as } b\leq 0)$$

$$\leq [c(j-1)] f_0[c(j-1)] \sum_{k=0}^{j-1} \frac{[ac(j-1)]^{k+1}}{(k+1)!} \leq [c(j-1)] f_0[c(j-1)]\, e^{ac(j-1)}.$$

As $f_0(z) = \frac{\sinh(z)}{z}$, we have that the summand is of exponential order.

Similar considerations clearly apply to the other summand in $v_j$. Thus $\{v_j\}$ has a non-trivial transform ie. we can find a $z_0>0$ such that $\bar{v}(z)=\sum_{j=0}^{\infty} z^j v_j$ converges for all $z, |z|<z_0$. All the properties of $f_k$ that have been used can be derived from the formulae for $f_k$ given on p444 of Abramowitz and Stegun (loc cit).

For $|z|<z_0$, we have $\bar{v}(z) = \sum_{j=0}^{\infty} z^j v_j = \sum_{j=1}^{\infty} z^j v_j$ (as $v_0 = 0$)

$$= \gamma_0 \sum_{j=1}^{\infty} z^j \sum_{k=1}^{j} \frac{(-a)^k}{k!} e^{b(j-k)} [c(j-k)]^{k+1} \{ f_{k-1}[c(j-k)] - f_k[c(j-k)] \}$$

$$= \gamma_0 \sum_{j=1}^{\infty} z^j \sum_{m=0}^{j-1} \frac{(-a)^{j-m}}{(j-m)!} e^{mb} (cm)^{j-m+1} \{ f_{j-m-1}(cm) - f_{j-m}(cm) \}$$

$$= \gamma_0 \sum_{m=0}^{\infty} \sum_{j=m+1}^{\infty} z^j \frac{(-a)^{j-m}}{(j-m)!} e^{mb} (cm)^{j-m+1} \{ f_{j-m-1}(cm) - f_{j-m}(cm) \}$$

$$= \gamma_0 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z^{m+n} \frac{(-a)^n}{n!} e^{mb} (cm)^{n+1} \{ f_{n-1}(cm) - f_n(cm) \}$$

$$= A + B$$

where $A = \gamma_0 \sum_{m=0}^{\infty} z^m e^{mb} \sum_{n=0}^{\infty} \frac{(-az)^n}{n!} (cm)^{n+1} \{ f_{n-1}(cm) - f_n(cm) \}$

and $B = -\gamma_0 \sum_{m=0}^{\infty} z^m e^{mb} (cm) \{ f_{-1}(cm) - f_0(cm) \}$

The inversion of the order of summation above requires justification: a proof can be constructed based essentially on the fact that a sequence of absolutely convergent numbers can be summed in any order.

As $(cm) f_{-1}(cm) = \cosh(cm)$ and $(cm) f_0(cm) = \sinh(cm)$, $B$ can be summed to $B = -\gamma_0 / 1 - z e^{(b-c)}$.

Also $\sum_{n=0}^{\infty} \frac{(-az)^n}{n!} (cm)^{n+1} f_{n-1}(cm) = \cosh[cm(1-2az)^{\frac{1}{2}}]$. This result is given on p445 of Abamowitz and Stegun (loc cit), and $\cosh[cm(1-2az)^{1/2}]$ is sometimes referred to as the generating function for $f_k$. Actually the result is nothing more than a development of $\cosh[cm(1-2az)^{\frac{1}{2}}]$ as a power series in $z$. See Watson (Theory of Bessel Functions, p140 Cambridge University Press) for details. Differentiation of the generating function also gives

$$\sum_{n=0}^{\infty} \frac{(-az)^n}{n!} (cm)^{n+1} f_n(cm) = \frac{\sinh[cm(1-2az)^{\frac{1}{2}}]}{(1-2az)^{\frac{1}{2}}}$$

These results then give

$$A = \frac{\gamma_0}{2} \left[1 - (1-2az)^{-\frac{1}{2}}\right]\left[\frac{1}{1-ze^{b+c(1-2az)^{\frac{1}{2}}}}\right] + \frac{\gamma_0}{2}\left[1 + (1-2az)^{-\frac{1}{2}}\right]\left[\frac{1}{1-ze^{b-c(1-2az)^{\frac{1}{2}}}}\right]$$

Similar considerations applied to $\{u_j\}$ give

$$\bar{u} = (-r_0 a_0) \times \frac{1}{2(1-2az)^{\frac{1}{2}}} \times \left[ \frac{1}{1-ze^{b+c(1-2az)^{\frac{1}{2}}}} - \frac{1}{1-ze^{b-c(1-2az)^{\frac{1}{2}}}} \right]$$

for small enough $|z|$.

As yet it has been unimportant to specify which branch of the square root we are using. For definiteness in the analysis which follows, however, we take $(1-2az)^{1/2}$ as that branch of the radical that assigns a positive root to $1-2az$ whenever $z$ is real and $< \frac{1}{2a}$.

We are interested in $\bar{q}(z) = \bar{v}(z) + \bar{u}(z) = A+B+\bar{u}$.

Some care must be taken in interpreting this equality.

The function $\bar{q}(z)$ on the l.h.s. represents a transform which is related to the transform $\bar{p}(z) = \sum_{j=0}^{\infty} z^j p_j$ where $p_j = p(y_j)$. By the r.h.s. we understand an explicit expression for $\bar{u}+A+B$ in terms of radicals etc. as above. The equality of l.h.s. and r.h.s. then means that for certain $z$, in fact $|z|$ small enough, the power series ~~of~~ $\bar{q}(z)$ converges and may be calculated by the expression $\bar{u}+A+B$. Let us denote $\bar{u}+A+B$ by $E$.

Then
$$E = \frac{(-r_0 a_0)}{2(1-2az)^{\frac{1}{2}}}\left[\frac{1}{1-z\,e^{b+c(1-2az)^{\frac{1}{2}}}} - \frac{1}{1-z\,e^{b-c(1-2az)^{\frac{1}{2}}}}\right]$$
$$+ \frac{\gamma_0}{2}\left[1-(1-2az)^{-\frac{1}{2}}\right]\left[\frac{1}{1-z\,e^{b+c(1-2az)^{\frac{1}{2}}}}\right] + \frac{\gamma_0}{2}\left[1+(1-2az)^{-\frac{1}{2}}\right]\left[\frac{1}{1-z\,e^{b-c(1-2az)^{\frac{1}{2}}}}\right]$$
$$-\gamma_0/1-ze^{b-c}.$$
We recall that $0 < a = \frac{2r_1 r_0}{(r_1+r_0)^2} < \frac{1}{2}$, $b = \frac{-(r_1-r_0)^2}{2r_1r_0} < 0$, $c = \frac{r_1^2-r_0^2}{2r_1r_0} > 0$.
Also it can be shown that $1-ze^{b+c(1-2az)^{1/2}}$ has no roots $z$, $|z| \leq 1$, real or complex, except $z=1$, which, moreover, gives a double root. (We mention that this result is related to the fact that the denominator of the transfer function $\bar{H}(s)$ has no zeros with real part $\geq 0$, except $s=0$ which (again) gives a double zero.) Clearly then $E$ is an analytic function of $z$ for $|z|<1$. Thus $\bar{q}(z)$, which equals $E$ for small enough $|z|$, can be continued to any $z$ with $|z|<1$. Now a power series can be continued right up to its first singularity. Hence $\bar{q}(z)$ converges for all $|z|<1$. Also by the uniqueness of the continuation we must have $\bar{q}(z) = E$ for all $|z|<1$.

Now $F(z) = \left[\frac{\gamma_0\left((1-2az)^{\frac{1}{2}}-1\right) - a_0 r_0}{1-z\,e^{b+c(1-2az)^{\frac{1}{2}}}}\right]$ is a combination of terms in $E$.
The numerator can be expanded as a Taylor series in a small disk about $z=1$ as $\gamma_0\left[(1-2az)^{\frac{1}{2}}-1\right] - a_0 r_0 = \frac{\gamma_0 a}{(1-2a)^{1/2}}(1-z) - \frac{\gamma_0 a^2 (1-2a)^{1/2}}{2(1-2a)^2}(1-z)^2 + \cdots$
Also we can expand $1-z\,e^{b+c(1-2az)^{1/2}} = \frac{1-a}{2(1-2a)}(z-1)^2 + \cdots$ for $z$ near 1.

Thus $(1-z)\,F(z)$ is analytic in a neighbourhood of 1. (we are defining $(1-z)\,F(z)$ at $z=1$ by continuity).

Clearly then all these results can be used to prove that $(1-z)\bar{q}$ is analytic in a disk about $z=0$ with radius $> 1$, where it is given, moreover, by $(1-z)E$. As $(1-z)\bar{q}$ is a continuous function within its radius of convergence we have, in particular, continuity at $z = 1$. Thus where $\{d_j\} = \{q_{j+1} - q_j\}$ is the first difference in $\{q_j\}$, we have proved the existence and continuity of its transform $\bar{d}(z) = (\frac{1}{z} - 1)\bar{q}(z) - \frac{q_0}{z}$ at $z = 1$. Hence we have justified the applicability of the final value theorem for $\{d_j\}$.

As $\gamma_0 e^{r_1 y} \xrightarrow[y]{\infty} 0$, we can use the theorem as

$$\lim_{j\to\infty} p(y_j) = \lim_{z\to 1}\left[\bar{q}(z)(1-z)\right] = \lim_{z\to 1} E(1-z) = \frac{\left[\frac{\gamma_0 a}{(1-2a)^{1/2}}\right]}{\left[\frac{1-a}{2(1-2a)}\right]} \times \frac{1}{2(1-2a)^{1/2}}$$

$= \frac{\gamma_0 a}{1-a} = \frac{-2r_1^2 r_0^2}{r_0^2 + r_1^2}$, which is the result already obtained for $\lim_{y\to\infty} p(y)$ in Section 2. through an (as yet unjustified) application of the final value theorem. As a technical point we mention that both $u_j$ and $v_j$ are $\sim j$ for large $j$. This behaviour should be important in a perturbation theory. In detail, if we work with

$\bar{u}(z) = \frac{(-r_0 a_0)}{2(1-2az)^{1/2}}\left[\frac{1}{1-z e^{b+c(1-2az)^{1/2}}} - \frac{1}{1-z e^{b-c(1-2az)^{1/2}}}\right]$ we see that $\bar{u}(z)(1-z)^2$ is analytic at $z = 1$. Then the second difference in $\{u_j\}$ (defined as $\{b_{j+1} - b_j\}$ where $\{b_j\} = \{u_{j+1} - u_j\}$ is the first difference in $\{u_j\}$ ) converges in a disk about $z = 0$ with radius $> 1$. Hence the final value theorem can be used to determine $b^0 = \lim_{j\to\infty} b_j$. Then by Cesaro $(\sum_{k=0}^{j} b_j)/j = (u_{j+1} - u_0)/j \xrightarrow[j]{\infty} b^0$ ie $u_j \sim j$. Similarly $v_j \sim j$. (The convergence theorem, in the form we are using it now, can be regarded as the complex analogue of a theorem in Titchmarsh ( The theory of Functions, 2nd edition p226, section 7.51: Oxford at the Clarendon Press).

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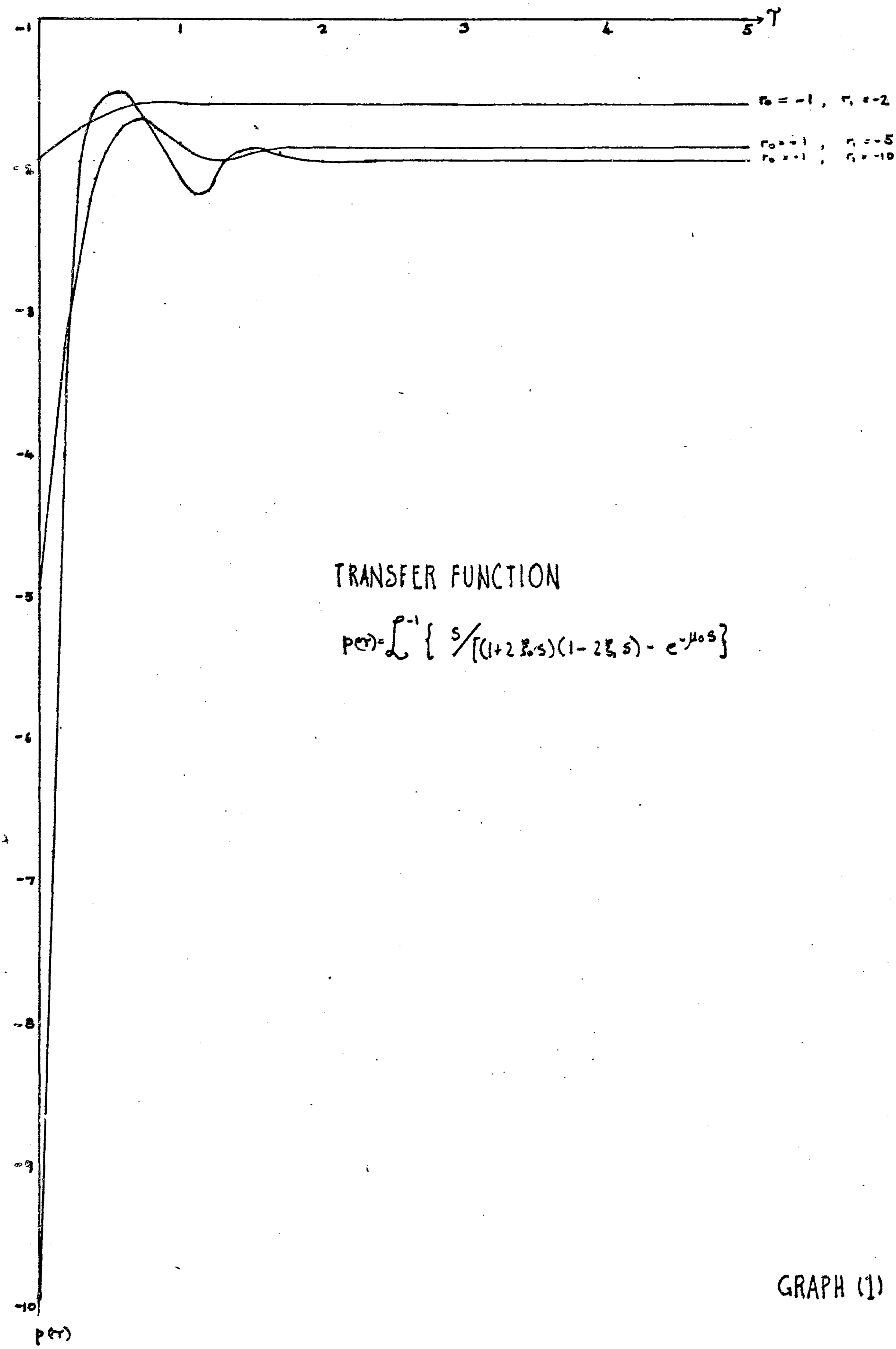
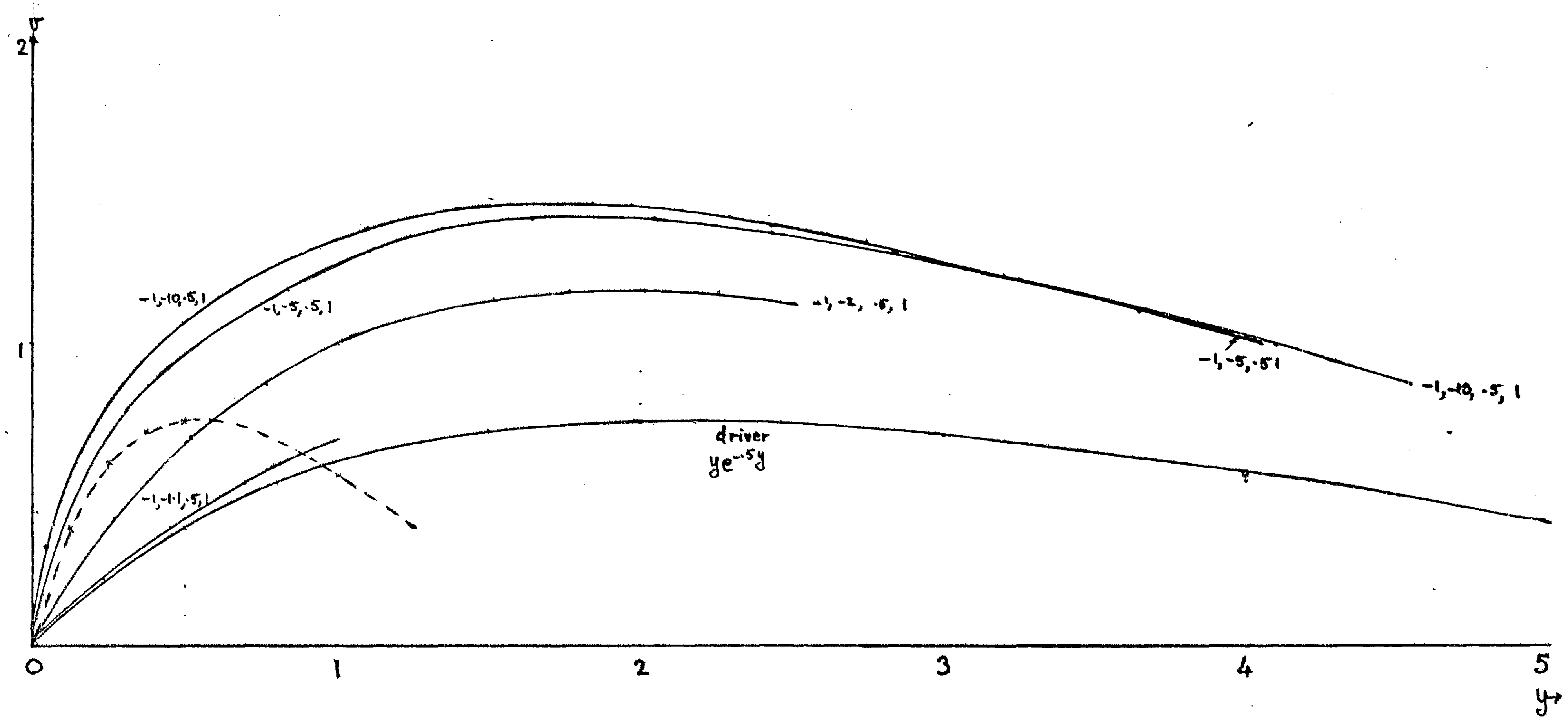
GRAPH (1) (Refer page 19).

Filter response p ($\tau$) vs $\tau$ for Iospheres of different dimensions $r_o$, $r_1$. The overshoot referred to on p.19, is largest for the Iosphere $r_o = -1$, $r_1 = -10$: in this case $x_1/x_o = r_1/r_o$ (cf fig p.13) is greatest. Evidently, in this case the light region $x > x_1$ (cf p.13) with density $\propto \beta^2 = (x_o/x_1)^4 << 1$ is most sensitive to motions in the dense ($\beta^2 = 1$) region $x < x_o$.


τ
1
2
3
4
5
-1
-3
-4
-5
-6
-7
-8
-9
-10
p(τ)
$r_0 = -1$ , $r_1 = -2$
$r_0 = -1$ , $r_1 = -5$
$r_0 = -1$ , $r_1 = -10$
TRANSFER FUNCTION
$p(\tau) = \mathcal{L}^{-1}\left\{ s \big/ \left[(1+2\xi_0 s)(1-2\xi_1 s) - e^{-\mu_0 s}\right] \right\}$

GRAPH (1)

GRAPH (2) (Refer text p.20). Response $v(y)$ at $x_1$ of the filter (cf fig p.13) to a gamma function driver $y\ e^{-.5y}$ at $x_o$ for Iosphere's of different dimensions $r_o$, $r_1$. (Refer to text p.20 for parameterization of curves).
The signal emerging from $x_1$ should be desteepened: this is seen by comparing the dotted curve (which represents a renormalization of $y\ e^{-.5y}$ appropriate for the specific speed $\beta_1$ at $x_1$ when $r_o = -1$, $r_1 = -2$) with the curve $(-1, -2, .5, 1)$, which is its measured response at $x_1$.



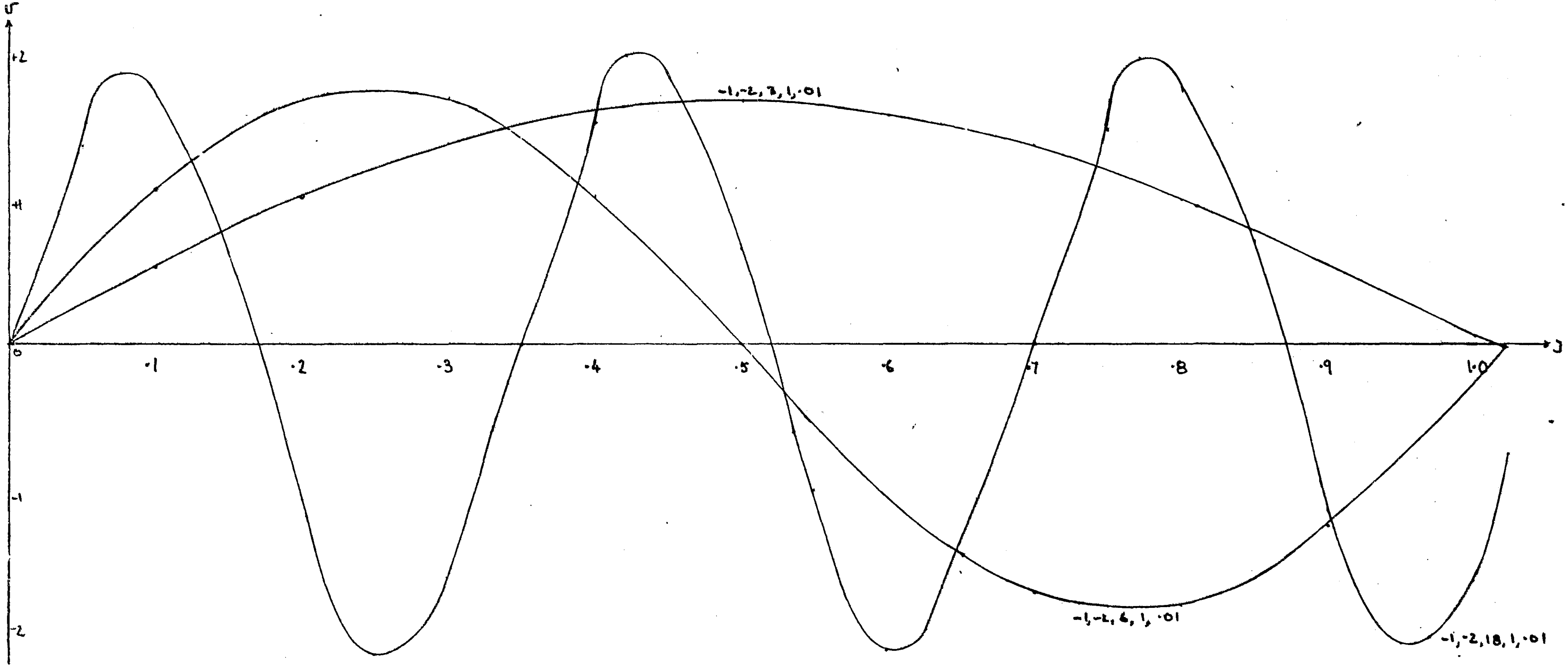

GRAPH (2)

GRAPHS (3) (Refer text p.20). Response $v(y)$ at $x_1$ of the filter (cf fig p.13) to sinusoid drivers $f(\xi_0 - y) = A \sin By$ (refer to text p.20 for parameterization of the curves) at the input $x = x_0$.

Evidently the response is capacitive eg. the frequency $B = 6$ (curve (-1, -2, 6, 1, .01)) excites an amplitude $\approx 1.8$, while the higher frequency $B = 18$ (curve -1, -2, 18, 1, .01) excites an amplitude $\approx 2.1 > 1.8$.

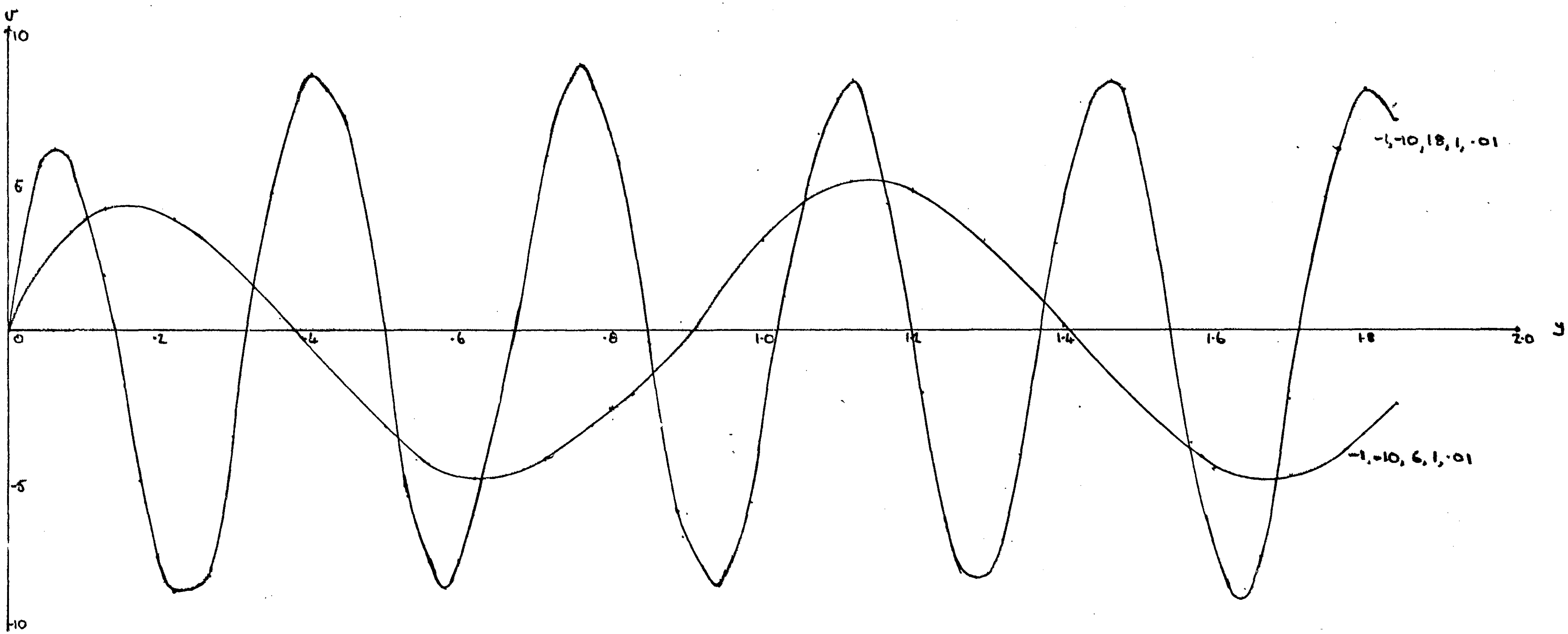
Then again the light region ( $\beta^2 = (x_0/x_1)^4 = (1/10)^4$, cf caption to Graph (1)) is more sensitive (curve (-1, -10, 18, 1, .01)) reaching an amplitude $\approx 8$ at the same frequency 18 as the region (-1, -2), (curve -1, -2, 18, 1, .01), reaches 2.1. These results are in agreement with the qualitative discussion preceding p.20.

RESPONSE TO SINUSOID




GRAPH (3)

RESPONSE TO SINUSOID



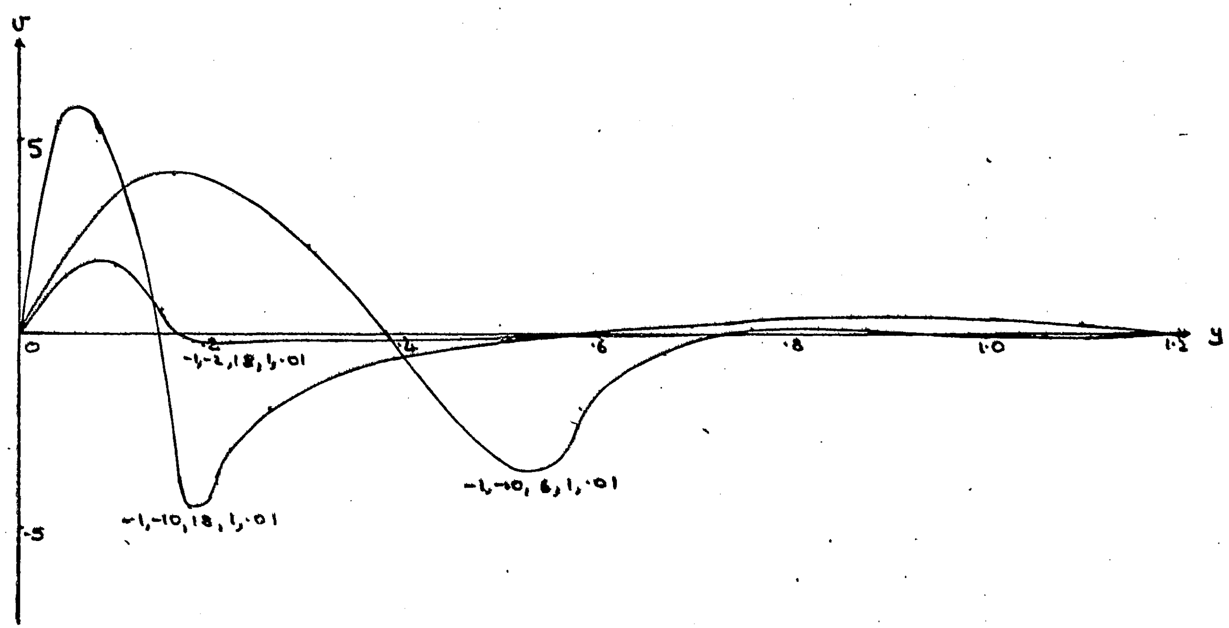

GRAPH (3)

GRAPH (4) (Refer text p.20). Response $v(y)$ at $x_1$ of the filter (cf fig p.13). to the single pulse

$$f(\xi_0 - y) = \{1 - u(y - \pi/B)\} \times \{A \sin By\} \text{ at } x_0.$$

( $u$ is the unit Heaviside). The frequency/density dependences described in the caption to Graph (3) are again in evidence eg. the response (-1, -10, 18, 1, .01) of the lighter filter, ( $\approx$ 6.0) is greater than the heavier (-1, -2, 18, 1, .01) which gives a response $\approx 2 < 6.0$ at the same frequency 18. Then again the filter (-1, -10) responds better ( $\approx$ 6.0) to the frequency 18 than to the lower frequency 6 (response $\approx 4.1 < 6.0$).

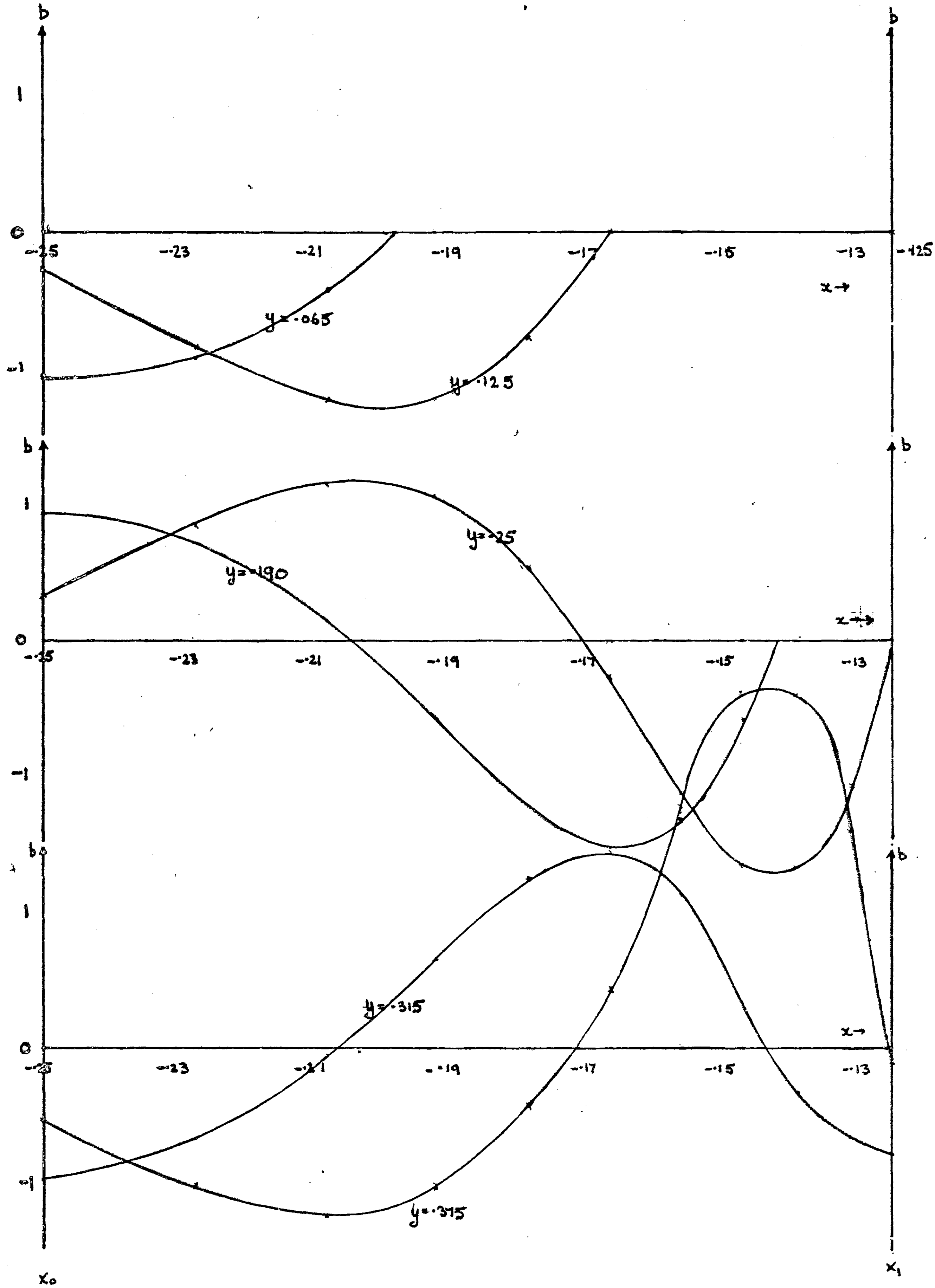
RESPONSE TO SINGLE PULSE



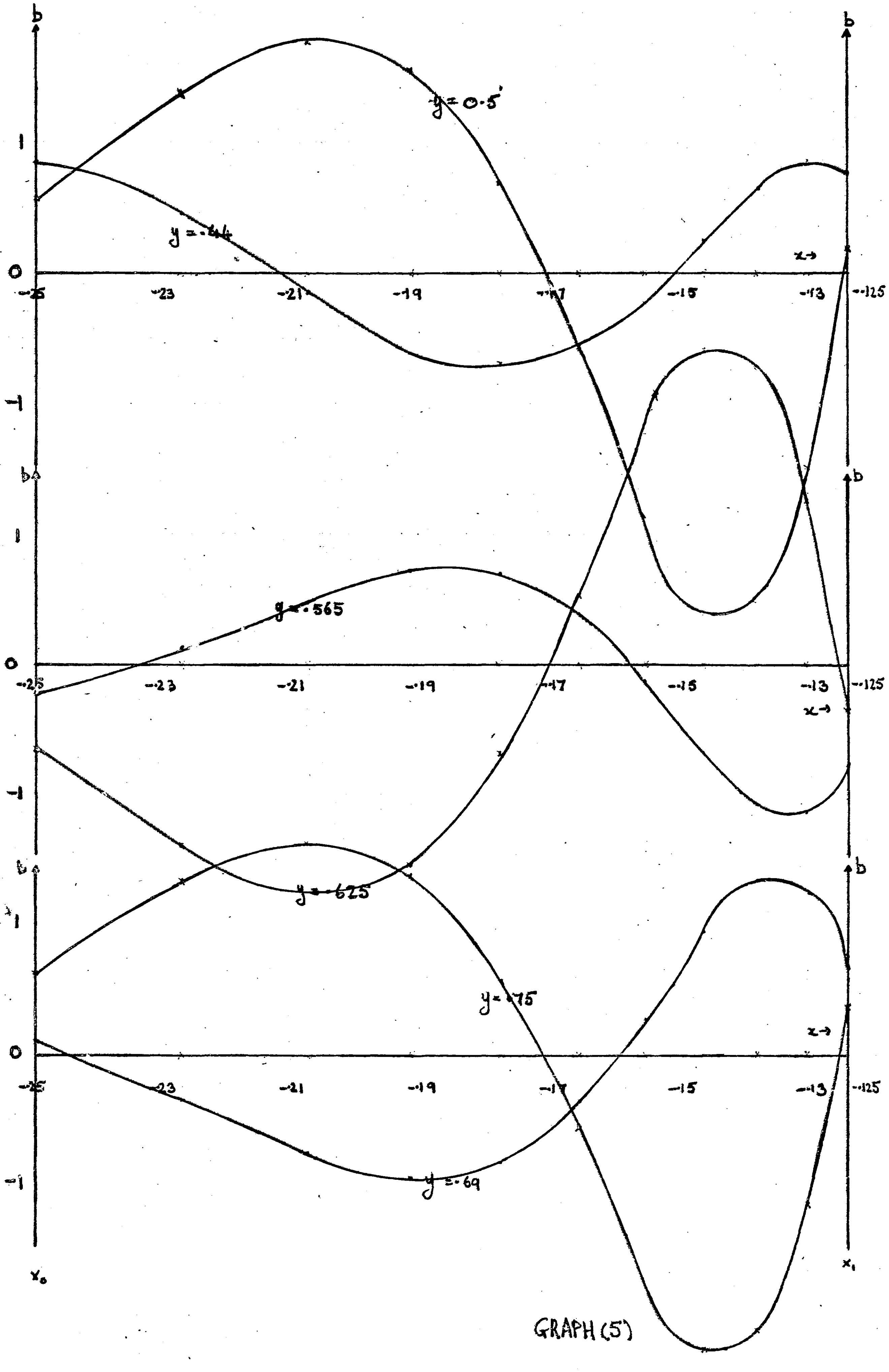

GRAPH (4)

GRAPHS (5) (Refer text pp.23-24). Response $b(x,y)$ of a disconnected (cf fig p.23) Ionosphere to a driver $v = f_o (y - x_c) \sin 24y$ impinging on $x_c$ from the left. Here $x_o = -\frac{1}{4}$, $x_1 = -1/8$, and the magnetic field distribution is plotted at successive time intervals $\Delta y \approx .0625$, which is a quarter of the travel time ($4 \times .0625 = .25$) for a signal from $x_o$ to $x_1$.

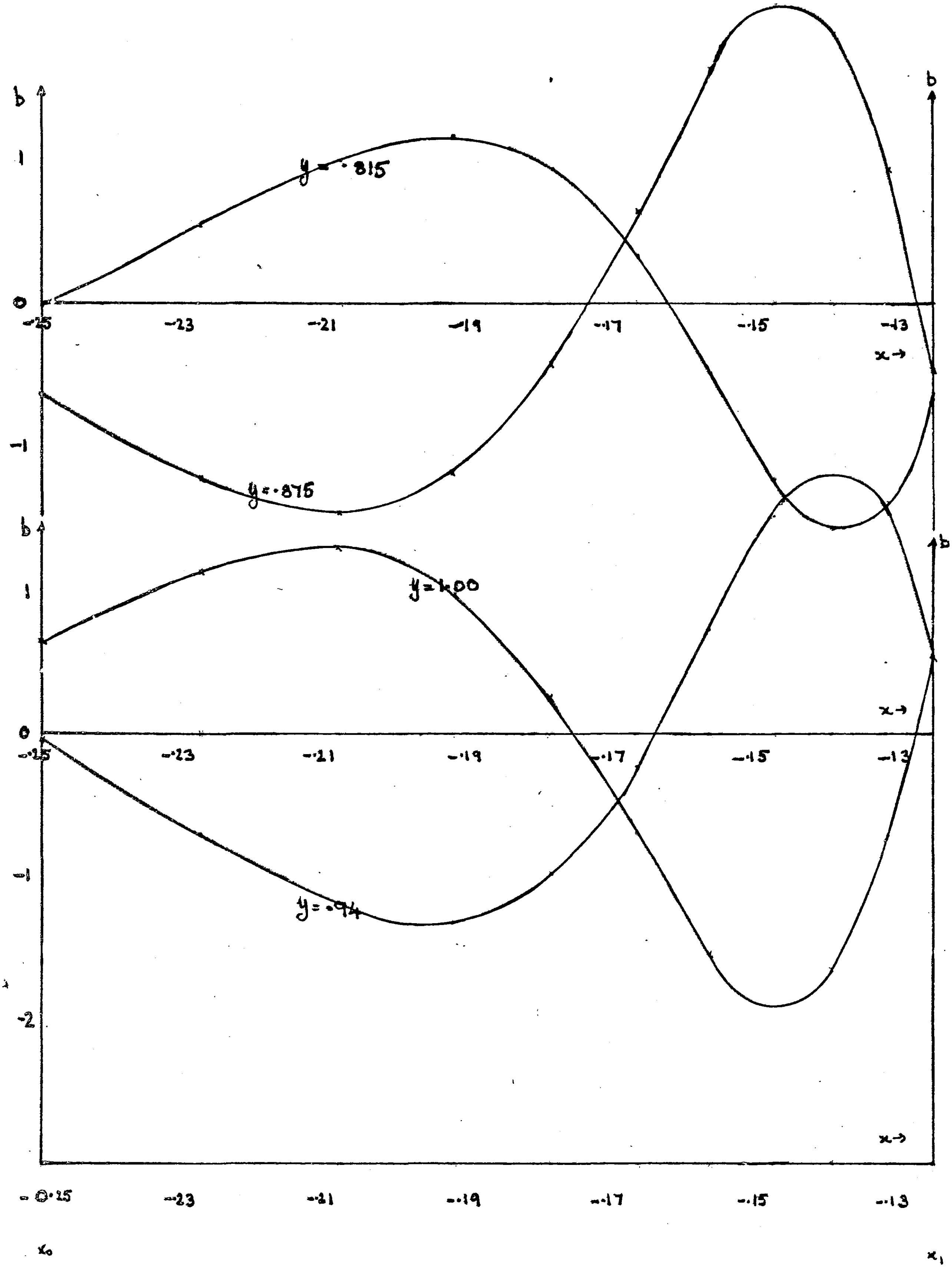
The signal steepens dramatically into the Ionosphere. At large times ($y > .5$) nodes and antinodes tend to develop at $x \approx (-.17, -.25)$ and $x \approx (-.14, -.20)$ respectively. The antinodes should be regarded as hot spots for possible instabilities feeding off b.




GRAPH (5)



GRAPH (5')

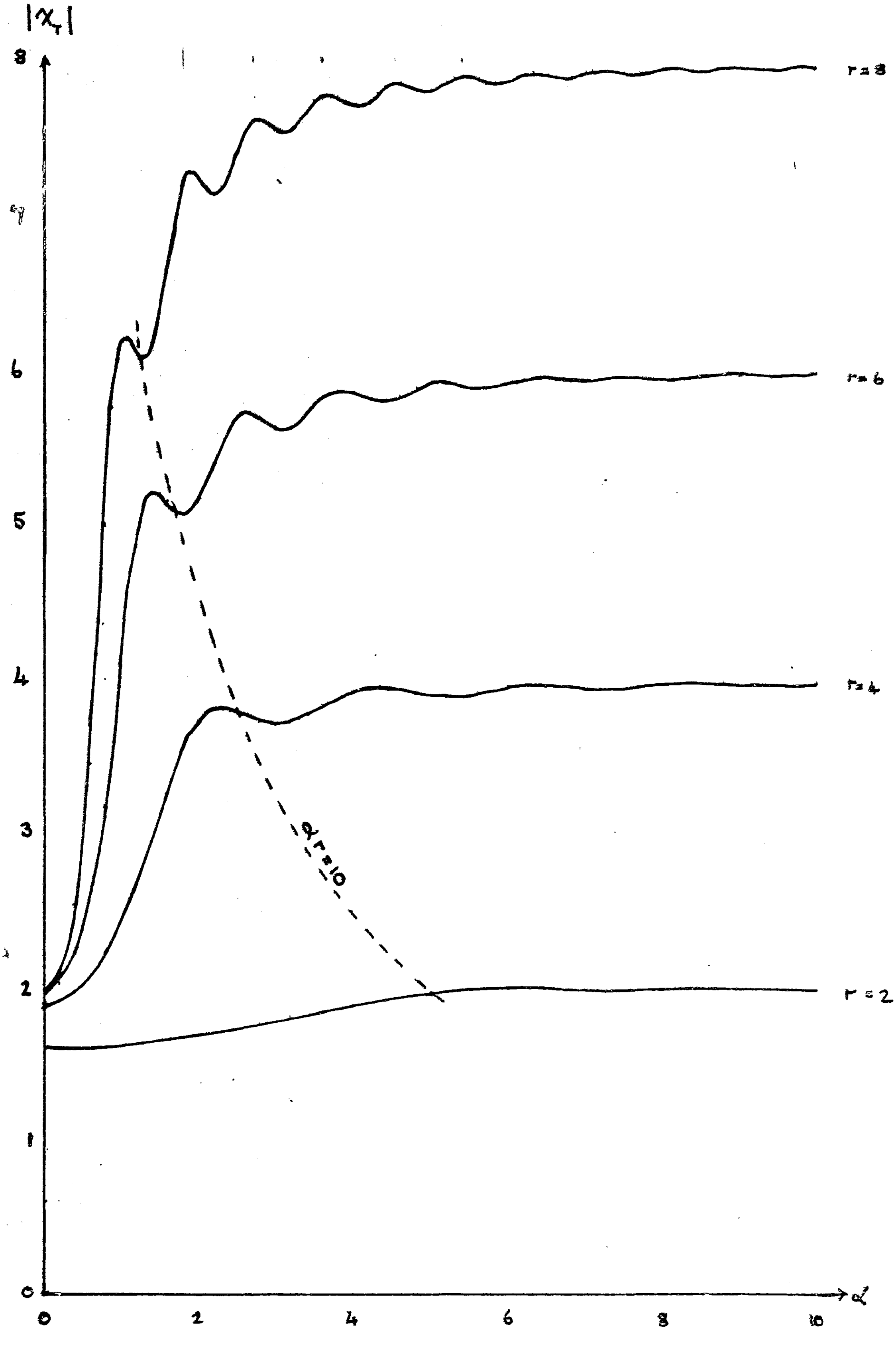

b
1
0
-1
y = ·815
y = ·875
-·25
-·23
-·21
-·19
-·17
-·15
-·13
x →
b
1
0
-1
-2
y = 1·00
y = ·94
x →
-·25
-·23
-·21
-·19
-·17
-·15
-·13
x →
- 0·25
-·23
-·21
-·19
-·17
-·15
-·13
$x_0$
$x_1$

GRAPH (5)

GRAPH (6) (Refer text pp. 30-31). Response $|\chi_1|$ at $x_1$ of a filter (cf top figure on p.30) to a sinusoidal exitation at $x_0$, vs $\alpha = -2\,\omega\beta_0 x_0$ (refer text p.31) for filters of dimensions $r = x_0/x_1 = 2, 4, 6, 8$.

The general increase of each curve with $\alpha$ results from the high frequency bias of the system (cf captions to GRAPHS (3) and (4)). The fluctuations on the curves are interference effects between $x_0$ and $x_1$, as in optical filter theory. (cf text p.31).

(The dashed line $\alpha r = 10$ is needed in the parametric analysis on p.32).




GRAPH (6)