INVESTIGATING RELATIONSHIPS BETWEEN MATHEMATICS TEACHERS' CONTENT KNOWLEDGE, THEIR PEDAGOGICAL KNOWLEDGE AND THEIR LEARNERS' ACHIEVEMENT IN TERMS OF FUNCTIONS AND GRAPHS.

by

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DECLARATION:

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ABSTRACT

This study used diagnostic tests, questionnaires and interviews to investigate explore teachers' subject content knowledge (SCK) and pedagogical subject knowledge (PCK). It also explored teachers' and learners' misconceptions within the topic of graphicacy and how teachers' SCK and PCK possibly affect learner achievement. A small sample of teachers were drawn from the Keiskammahoek region; a deep rural area of the Eastern Cape. These teachers were part of the Nelson Mandela Metropolitan University (NMMU) Amathole Cluster Schools Project who were registered for a three-year BEd (FET) in-service programme in mathematics education. As part of the programme they studied mathematics 1 and 2 at university level and received quarterly non-formal workshops on teaching mathematics at FET level. The findings of this study suggest that teachers with insufficient SCK will probably have limited PCK, although the two are not entirely dependent on each other. In cases where teachers' displayed low levels of SCK and PCK, their learners were more likely to perform poorly and their results often indicated similar misconceptions as displayed by their teachers. This implies that we have to look at what teachers know and what they need to know in terms of SCK and PCK if we are to plan effectively for effective teacher development aimed at improving learner performance.

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CHAPTER ONE

INTRODUCTION AND OVERVIEW

1. INTRODUCTION

Research in mathematics education has identified a number of common errors and misconceptions in learners' graphical thinking (Bell, Brekke & Swan, 1987; Clement, 1985; Hadjidemetriou & Williams, 2001, 2002). Teacher awareness of these misconceptions is considered by many researchers as an invaluable tool to help reformulate learners' understandings in order for them to use graphs more meaningfully (Cornu, 1991; Even, 1998). Teacher awareness of these misconceptions and errors as experienced by their learners means their knowledge surpasses mere subject and content knowledge. This awareness is defined by Shulman, (1986) as Pedagogical Content Knowledge (PCK).

This study investigated teachers' PCK by means of a questionnaire based on the work of Hadjidemetriou and Williams (2001, 2002) and an interview protocol which required them to predict how difficult their children would find items on a diagnostic test. They were also asked to suggest likely errors and misconceptions that they might expect their children to make and to suggest methods/ideas they would use to help learners overcome these difficulties. The study also used a diagnostic assessment instrument developed from literature on children's learning and misconceptions (Hadjidemetriou & Williams, 2001, 2002) to investigate the graphical understandings and misconceptions of these teachers' grade eleven learners in a sample of rural schools in the Eastern Cape, South Africa.

2. THEORETICAL FRAMEWORK

Research has shown many misconceptions and errors in thinking when testing learners on functions and graphs. Research by Clement (1985), Hadjidemetriou and Williams (2001, 2002), Kerslake (1981) and Leinhardt, Zaslavsky and Stein. (1990) specifically highlights the difficulties experienced by students when working with functions and graphs. These include slope-height confusion, being unable to distinguish between the slope and the highest value, linearity prototype, sketching straight lines when one is not supposed to, x=y prototype, believing that all graphs have a gradient of 1, origin prototype, all graphs must go through the origin, graphs as a picture, the inability to see that the graphs represents a relationship, coordinates, reversing x and y values and being unable to adjust their knowledge from one situation to another. Van Dyke and White (2004) also looked at problem areas associated with the difficulties encountered when learning graphs which include students not knowing which aspects of the graph to focus on; correctly reading a graph; identifying pertinent detail; reading graphs in given contexts; understanding the Cartesian plane; knowing the difference between a constant function and a function with a constant rate of change; and using functional notation.

If these difficulties and problems are experienced by learners it is possible that their teachers experience the same misconceptions as a result of their educational background and experiences. In their research, Leinhardt, Zaslavsky and Stein (1990; p, 45), reviewed literature on graphs and functions and claimed that, “of the many articles we reviewed almost 75% had the obligatory section at the end called something like ‘Implications for teaching’ but few dealt directly with research on the study of teaching these topics.” This implies that teachers need to participate in interventions to keep them abreast with, and aware of, new ideas and approaches that assist learners in the difficulties mentioned in the paragraphs above. Hadjidemetriou and

Williams (2002) found that in some cases teachers misunderstood questions and displayed a limited understanding of certain concepts. The teachers in their study also overestimated the difficulty on certain questions because they could not envision an answer without using sophisticated knowledge of gradients when this knowledge was not necessary (Hadjidemetriou & Williams, 2002).

3. STATEMENT OF THE PROBLEM

South African education has faced many changes in the last decade and implementation of the new curriculum in the FET phase poses many questions, problems and difficulties (Jansen, 1999; Taylor & Vinjevold, 1999). Not the least of these is Learning Outcome 2 (Functions and Algebra) of the National Curriculum Statement (NCS), which requires a learner to be able to "investigate, analyse describe and represent a wide range of functions and solve related problems" (Department of Education, 2003; p 12). These activities require a fairly sophisticated understanding of functions and their graphical representations, but little is said of the many universal difficulties in terms of understanding and using functions and graphs as highlighted by Clement (1985), Hadjidemetriou and Williams (2001), Kerslake (1993), Leinhardt, et al. (1990) and others.

Working with teachers in the Keiskammahoek area in the Eastern Cape raised questions in terms of these teachers understandings of functions and graphical representations, their pedagogic content knowledge (in terms of being aware of common difficulties experienced by learners when working with functions and graphs) and whether their misconceptions and suspected low level of pedagogical content knowledge can be correlated with their learners' achievement in terms of using functions and graphs.

4. RESEARCH QUESTIONS

The primary question of this study focuses on the teachers' content knowledge and PCK in terms of Functions and Graphs. Therefore the primary research question is:

Do relationships exists between teacher's level of content knowledge, their PCK and their misconceptions about functions and graphs and their learner's achievement in terms of using graphs and, if there are, what is the nature of these relationships?

The secondary questions which must be answered in order to answer the primary question are:

- *What is the level of content knowledge held by their learners?*
- *What is the level of content knowledge held by the teachers?*
- *What common misconceptions in terms of graphs are held by the teachers and their learners?*
- *What is the level of pedagogical content knowledge held by the teachers?*

5. RESEARCH METHOD

The research involved both qualitative and quantitative methods in evaluating the content and pedagogical knowledge of the teachers and learners in the study. These methods include:

- Diagnostic testing of the learners' content knowledge.
- Diagnostic testing of teachers' content knowledge.
- Teacher questionnaires on their perceptions of the level of difficulty that their learners will encounter when answering question on functions and graphs.

- Teacher Interviews.

The teachers were required to complete the diagnostic test to assess their content knowledge and misconceptions. A questionnaire and interview schedule, which investigated their PCK in terms of identifying problems their learners might have with certain questions on the test as well as suggestions as to how to address these misconceptions, were developed from the teacher diagnostic test. The learners were then given the test to complete. An attempt was made to treat the data statistically in order to determine whether any correlations exist between teacher's predictions and the learners' actual achievement, but the sample sizes were too small to yield statistically significant results and this route was not pursued. The questionnaire, interview and test data were triangulated in an attempt to answer the research questions.

6. SAMPLE AND SETTING

The sample used in this study was a convenience sample consisting of five teachers and 66 of their learners (their grade 11 classes) who took part in the Nelson Mandela Metropolitan University (NMMU) Amathole Cluster Schools Project in the Keiskammahoek region of the Eastern Cape. This initiative included registering the teachers on a three-year BEd (FET) in-service programme in mathematics and providing them with quarterly non-formal workshops on teaching mathematics at FET level. The BEd programme included first year and a semester of second year 'pure' mathematics, as well as modules focusing on the teaching of mathematics at FET level. The testing and interview aspects of this study took place during the second year of the programme.

The Keiskammahoek region is a deep rural area of the former Ciskei where most inhabitants live in huts without water-borne sanitation or electricity. The Ciskei was a 'self-governing homeland' during the apartheid era of South African history. The participating schools

are considered ‘previously disadvantaged’ due to the uneven support systems of the disparate educational systems that operated during this time and which were based on racial classification. These schools still have had limited access to educational resources and support from the current Department of Education in the Eastern Cape. The class sizes in mathematics at the time of the study were small as, prior to the intervention, no learners were registered for Mathematics Higher Grade in any of the participating schools, nor had a higher grade pass in mathematics ever been recorded in the Keiskammahoek region. The final year of the intervention took place in the same year that mathematics higher and standard grade were offered as part of the national curriculum to learners in South Africa, thereafter all grade 10-12 learners had to study either mathematics or mathematical literacy as compulsory subjects. The group of teachers and their learners were a convenience sample as the teachers were the participants in the Amathole Cluster Project and, as such, were accessible for the research project.

7. DATA COLLECTION INSTRUMENTS AND ANALYSIS

The diagnostic test for learners used in this study is based on the tests designed by Hadjidemetriou and Williams (2001) and investigates common graphical misconceptions held by high school learners. The test comprises of two sections of eight and six questions each (appendix A). Part 1 investigates graphicacy in various contexts, whereas Part 2 simply investigates knowledge of graphicacy competencies without a supporting context, as was done by Hadjidemetriou and Williams (2001).

The teachers were also required to complete the learner test before they were asked to answer the questionnaire designed to elicit their perceptions of the difficulties their learners would experience with the questions in the diagnostic test. The questionnaire also required the teachers to explain what they would do in order to help their learners to change their

misconceptions and how they would go about assisting their learners to cope with the difficulties and challenges posed by the questions. The data generated by the teachers completing the learner test gave an indication of their content knowledge and their level of understanding of the concepts involved, while the questionnaire provided insights into their pedagogical content knowledge. An interview schedule informed by the data generated by the diagnostic test and questionnaire was developed in order to further probe the teachers thinking in terms of their own understanding of the concepts and their perceptions of the difficulties that their learners would encounter.

8. ETHICAL ISSUES

The principles of informed consent were adhered to in this study and all teachers and learners' participation was voluntary. All of the teachers chose to be part of the study and persisted with the process until its completion. The teachers provided the necessary consent for the learners to participate in their capacity of acting *in loco parenti*s while the children were in school. The participants were informed that they would have access to the results and findings of the study and that their anonymity would be respected. These aspects were all covered verbally during the project contact sessions

9. RELEVANCE OF THIS STUDY

The South African curriculum focuses on applying mathematical concepts to real-life situations and multicultural approaches, whilst other countries focus on the understanding of concepts and the mastering of basic skills and principles (Reddy, 2006). This implies a change from the emphasis of merely sketching graphs and mechanically applying algorithms to deeper interpretation of functions and graphs. Learning Outcome 2: Functions and Algebra of the new National Curriculum Statement for Mathematics states that:

> A fundamental aspect of this outcome is that it provides learners with versatile and powerful tools for understanding their world while giving them access to the strength and beauty of mathematical structure….
>
> … to study the nature of the relationship between specific variables in a situation…
>
> … provides learners with models to describe and analyse such situations. The emphasis is on the objective of solving problems and not on the mastery of isolated skills (such as factorisation) for their own sake.
>
> (Department of Education, 2003; 12 & 13)

Reddy's analysis of the Trends in International Mathematics and Science Study of 2003 highlighted the fact that South African teachers were the least qualified in comparison to the rest of the sample of international teachers who participated in this study worldwide (Reddy, 2006). Norman (1993) believes that there is not enough written about the knowledge of teachers, whilst Williams (1993) states that the study of graphs with respect to "informing teaching and learning is in its infancy" (p. 314). In their studies of teachers PCK, Hadjidemetriou and Williams (2001, 2002) found that teachers held many misconceptions pertaining to graphs, indicating a weakness in their subject content knowledge. Teachers also struggled to accurately describe the level of difficulty with which the learners encountered certain graphing aspects of questions, often overestimating the intricacy of interpretative questions.

The purpose of the Amathole project was to improve mathematics results in the participating schools in the Keiskammahoek area. This was done by working with teachers in an attempt to improve their content knowledge and their understanding of how learners learn in general. Investigation of possible relationships between teachers' subject content knowledge

(SCK), PCK and the identification of common misconceptions related to learner achievement in graphical contexts, could contribute to the debate around enhancing teacher training methods (both pre- and in-service) and give greater insight into possible education models and strategies that could be employed.

10. OUTLINE OF THE STUDY

In this chapter I have discussed the reasoning behind the research, its relevance and its context as well as introducing the primary and subsidiary research questions. The methods used, sample and setting and the data collecting instruments are briefly described, as was recognition of the ethical issues involved in the study.

Chapter two provides a comprehensive literature review of teacher's PCK and the misconceptions encountered by learners and teachers alike with respect to the understanding and teaching of graphs. A critical analysis of recommendations made by previous research and the implications of these recommendations is discussed. In chapter three the research design is outlined and the use of the various methods employed is motivated. A description of how data was generated and analysed is also included. In chapter four the data generated by the research is presented. In chapter five the research findings are interpreted and the misconceptions of the participating teachers, and the impact that these understandings have on their teaching practice, are considered. The most important findings are highlighted in chapter six, implications for teacher development are noted, and recommendations for further research are made.

CHAPTER TWO

LITERATURE REVIEW

1. INTRODUCTION

In this chapter mathematics education, the requirements for promoting learner achievement, and understanding of mathematics, are briefly considered. The level of knowledge that mathematics teachers need to be considered proficient is noted, as are Fennema and Loef's (1992) four components of teacher knowledge. These components are knowledge of mathematics, knowledge of mathematical representation, teachers' knowledge of students and general knowledge of teaching and decision making. These four concepts fall broadly in two main themes, viz. content knowledge and pedagogical content knowledge. The first two concepts fall under the umbrella of subject content knowledge while the latter two are established under pedagogical content knowledge (Fennema & Loef, 1992). Misconceptions in mathematics are considered in general before entering into a discussion about misconceptions specifically pertaining to graphicacy. The issues mentioned above are then examined in terms of the new South African curriculum.

2. UNDERSTANDING AND MATHEMATICS

Kilpatrick, Swafford and Findell (2001) believe that an essential and critical component for proficiency in mathematics learning is the ability to understand. Their notion of understanding involves knowing more than just isolated facts, but rather that the learner has an

organised knowledge system and is able to connect new ideas to pre-existing theory. Swan (2001) suggests that understanding needs to be developed organically to give learners the opportunity to make sense of things in a frequently changing world. The belief being that understanding takes place when one creates and shapes conceptual frameworks. These frameworks lead to the development of links and multiple perspectives where a concept does not remain isolated as a single perception, but embraces many things and links to many different experiences (Swan, 2001).

Carpenter and Lehrer (1999) also advocate that understanding means that learning has to be generative and that understanding emerges and develops. They suggest that when someone understands aspects of mathematics they can apply this knowledge to learn new topics and solve new problems. Non-understanding on the other hand suggests that knowledge is compartmentalised and with isolated skills one is unable to solve problems unless they are explicitly covered by instructions. Although individuals learn in different ways Carpenter and Lehrer (1999) suggest that understanding, which is continuous, is developed in five general stages (Figure 2.1).

Understanding is the sense of achievement in being able to attain some order of merging ideas, seeing underlying structures and capturing of an idea (Sierpinska, 1994). Sierpinska records four mental operations which she believes are involved in understanding, viz.:

- Identification – we can recognise and focus our attention on a concept, while naming and describing it.
- Discrimination – we can note the resemblances and differences between the concept and others.

- Generalisation – in specific cases one can see the general properties of a concept.

- Synthesis – we can recognise unifying principles.

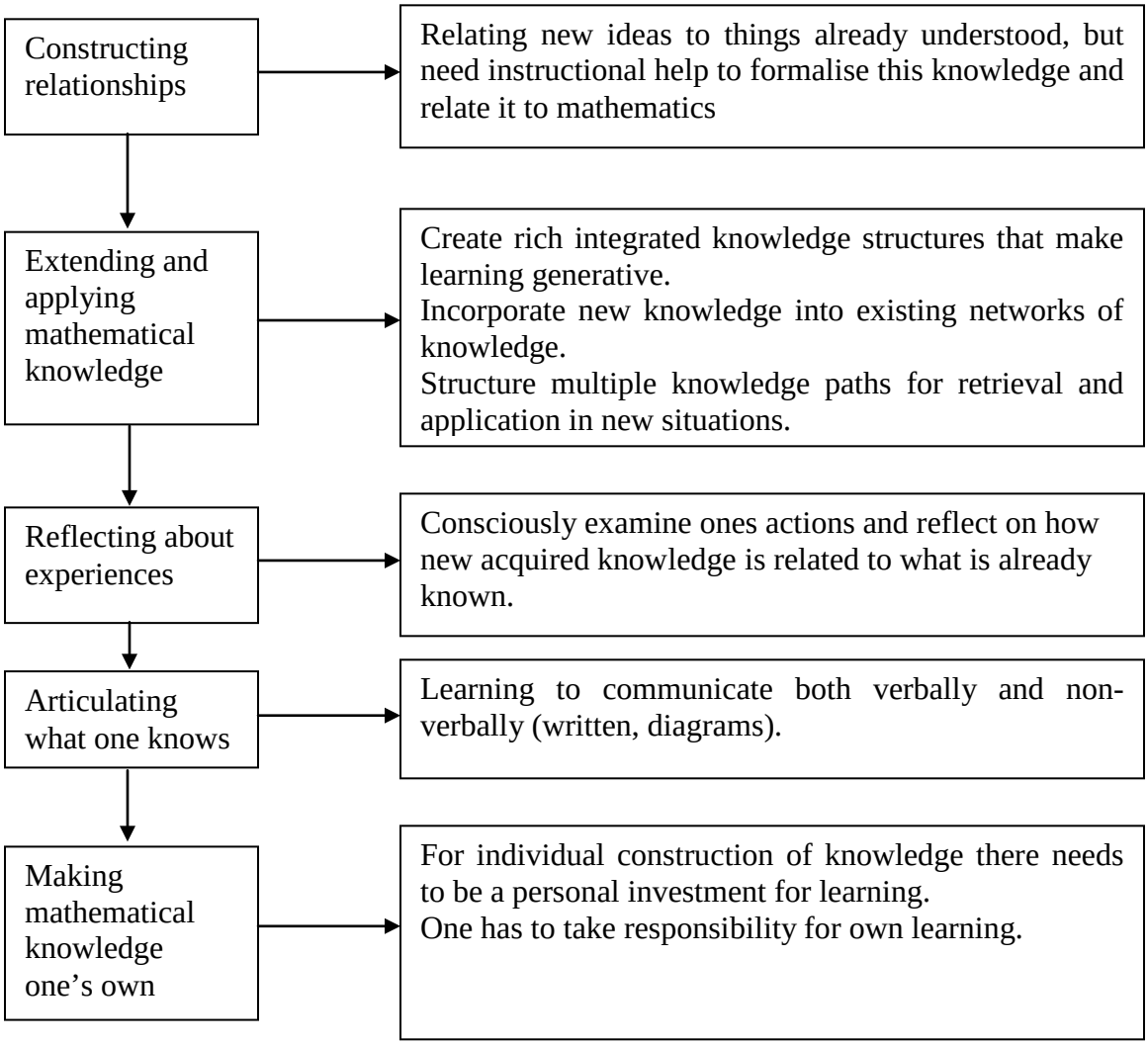


Figure 2.1: Carpenter and Lehrer (1999) five-stage model of understanding.

It is within this framework of understanding that I consider what a proficient teacher of mathematics needs to know.

3. PERCEPTIONS OF WHAT MATHEMATICS TEACHERS NEED TO KNOW

Ball, Hill and Bass (2005) propose that the quality teaching is reliant on teachers' subject content knowledge (SCK). They believe that sufficient mathematical proficiency is not being developed in the USA and, as a result, teachers cannot be expected to teach effectively. Ball, et al. (2005) believe that the deficiency is due to a self-perpetuating system fuelled by the fact that teachers themselves have been through the same system they are trying to improve, and therefore are unable to appreciate and create constructive mathematical learning experiences.

Numerous proposals have been put forward to improve teaching and learning in mathematics, but Ball, et al. (2005) warn that the effects of many of these have not been sufficiently rigorously researched to attest to their efficacy and some are even hotly contested. Suggestions for improving teaching and learning range from focusing on specific mathematical studies through coursework; majoring in subject-matter; taking a practice-grounded approach; training teachers in the specific mathematics they will use; to selecting prospective teachers from colleges that have shown over a long period of time that their teachers have the required levels of intelligence and mathematical competence. However, the missing common factor in these approaches remains: What mathematical knowledge is required for effective teaching?

Hill, Schilling and Ball (2004) note that while a measure of agreement as to what knowledge is required for teaching mathematics is probably vital, there is no consensus and mathematics educators are still grappling with the construct of what is "mathematics knowledge for teaching" (Hill, Schilling & Ball, 2004: 12). After reviewing approximately 30 years of research, Da Ponte and Chapman (2006) also acknowledge that there is still a need to ascertain distinct criteria of what knowledge mathematics teachers need to be considered proficient. Similarly, designing an assessment instrument to measure this proficiency is also seen as important. Hill, et al., (2004) suggest that the three most common strands of assessment for this

purpose are assessment of individuals' ability to solve mathematics problems; their ability to construct tasks for learners; and their ability to understand and apply SCK.

Stein, Baxter and Leinhardt (1990) reflect that while SCK is generally accepted as being a necessary condition for the development of PCK, exactly what SCK is needed to best develop PCK remains elusive. Ebert (1993) agrees but adds that while SCK is an important source which contributes to PCK development, it is not the only source, and showed a strong connection between the beliefs student teachers have about mathematics, their learners and how their learners learn and their PCK.

Ball and McDiarmid (1990) note that while it is obvious that teachers need to know what they teach, they agree that there is discord as to what SCK is required for effective teaching. However, they point out that if teachers do not have a sufficient level of SCK they run the risk of teaching inaccurately and failing to iron out misconceptions. As such, Ball and McDiarmid (1990) firmly believe that SCK affects teacher practices and feel that that merely asking the question 'What mathematics should prospective teachers study?'' will probably impact positively on the design of programmes aimed at improving teacher practice.

Finally, Stein, et al., (1990) suggest an increase in SCK preparation might impact on teaching practices in the classroom simply due to teacher confidence in mathematical proficiency, especially in terms of understanding learner misconception as an aspect of teaching. In terms of learner misconceptions, Barkai, Tsamir, Tirosh and Dreyfus (2002), Sánchez and Llinares (1992), Shriki and David (2001), and Graeber, Tirosh and Glover (1986) suggest that teacher training should be developed around strategies whereby teachers monitor and manage the impact of misconceptions on their own and their learners thinking. They recommend that

teachers need experience in connecting conceptual and procedural knowledge and increased in-depth SCK to enable them to do this.

The above varying perceptions of what mathematics teachers need to know, why they need to know it, and the fact that there is no commonly accepted understanding of what specific mathematical subject content knowledge effective teachers require, suggests a closer look at the issues of subject content knowledge, pedagogical content knowledge and learner misconceptions; issues which are important in this study which investigates teachers' PCK and their and their learner's misconceptions.

4. SUBJECT CONTENT KNOWLEDGE

As noted earlier, teachers' subject content knowledge is one essential category of the teachers' knowledge and it appears self-evident that a teacher should understand the content of the subject they are teaching (Aubrey, 1997; Ball & McDiarmid, 1990). Cramer (2004) highlights that teachers are often embarrassed by their lack of mathematical knowledge and are often resistant to the idea of their subject content knowledge being assessed; a necessary requirement of facilitating teachers' growth in content knowledge and in teacher training.

Da Ponte and Chapman (2006:462) refer to subject content knowledge as a "formal and refined field of academic discipline which is a critical attribute of a mathematics teacher". Leinhardt, Putman, Stein & Baxster (1991) point out that subject matter content (knowledge) includes concepts, algorithms, operation, connection, subsets and number system displayed in teaching a particular school-level curriculum. Shulman (1986:9) propose that SCK means teachers should "be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions".

The President's Education Initiative (PEI) research studies done in South Africa repeatedly found a lack of teacher content knowledge (Taylor & Vinjevold, 1999). Lessons presented by these teachers exposed their poor grasp of conceptual knowledge and revealed blatant errors in their teachings. Irwin and Britt (1999) argue that if teachers have a broad and integral subject content knowledge they are more likely to understand the true nature of mathematics and employ teaching approaches which will emphasise problem solving and enable learners to develop a more constructive and appropriate knowledge of mathematics. Consequently, teachers who have a better subject content knowledge show more flexibility in their teaching (Irwin & Britt, 1999).

Many studies have drawn attention to specific deficiencies in teachers' subject content knowledge. Linchevsky and Vinner (1989) found most of the expected misconceptions and misunderstanding of fractions that they anticipated from work with children were held by teachers'. Sánchez and Llinares (1992) found that pre-service elementary teachers were unable to identify the notion of unity when working with fractions greater than one. Graeber, Tirosh and Glover (1986) found an increase in teacher error when problems did not satisfy pre-existing models in word problems containing the operations of division and multiplication, while research by Pinto and Tall (1996) on rational numbers revealed that many teachers gave prescribed definitions that contained distortions while some teachers were unable to give a definition at all. Of those who could provide a definition, none used the definition as a tool for developing the meaning of the concept.

Hershkowitz and Vinner (1984) compared learner's knowledge and pre- and in-service teacher knowledge and found that teachers lacked basic geometrical subject knowledge, proficiency and analytical thinking capability. Ponte's (1985) study of subject content knowledge

and functions revealed that the participating teachers felt uneasy about working with geometrical data and found it difficult to understand the relationship between graphical and numerical information. Even (1990) found that many teachers did not have an understanding of the concepts of inverse, exponential, logarithmic, power and root functions. Thomas (2003) found many conflicting ideas on what a function was while Shriki and David (2001) also found that teachers provided a range of definitions relating to the parabola concept, indicating difficulties and misconceptions. On the concept of limit and continuity, Mastorides and Zachariades (2004) found clear gaps in teacher's understanding of this topic.

Ball and McDiarmid (1990) considered the different avenues where teachers learn their SCK. Their first encounter of SCK is usually during their school years, prior to any tertiary education. A second encounter takes place when they undergo professional training. However, it is a matter of debate as to how much SCK is actually addressed within the training of most mathematics teachers. The third and final encounter is through experiences in the classroom and the use of textbooks, which are at times erroneous. They refer to three facets when studying subject matter: firstly there is substantive knowledge, i.e. specific data, notions and topics relating to the respective subject studied. Secondly there is the knowledge about the subject, which is rarely explicit, but they are the ideas on which one builds as one encounters the subject more deeply. Lastly there is a disposition towards the subject, the likes and dislikes of certain topics in the subject (Ball & McDiarmid, 1990). Their knowledge of how to teach mathematics (pedagogical content knowledge) rests on the foundation of the SCK they have developed.

5. PEDAGOGICAL CONTENT KNOWLEDGE

Jones (2004) believes that teachers' who have good pedagogical content knowledge will be able to understand what their learners know and what they misunderstand, aspects that are

necessary to enable them to identify productive strategies for meaningful teaching in their classrooms. French (2002) suggests that many teachers introduce new topics of work by demonstrating 'worked examples', reinforcing the demonstrated procedures through repetition and extensive practice exercises, but argues that these exercises are often far removed from those that could lead to meaningful and purposeful development of sound algebraic ideas. Webb, et al's. (1998) PEI study supports the contention that improving teacher knowledge does improve learner achievement in mathematics. Adler (2002) agrees but argues that although there is no contention that teachers require subject knowledge to teach effectively, tension does arise when considering how to integrate SCK with how to teach the subject (PCK). Shulman (1987) underscores the fact that that subject knowledge in itself is not sufficient, and highlights the necessity of pedagogical content knowledge for making a difference in learner achievement.

Schön (1983) defines teacher knowledge as more than knowing facts, properties and relationships but also the identifying and solving professional problems in order to build learner knowledge. Ebert (1993) notes that pedagogical knowledge can be present without subject matter knowledge, and that subject knowledge enables subject teaching, but argues that both on their own are necessary but not sufficient conditions for effective teaching. Ball, Hill and Bass (2005) highlight that the mathematical knowledge required by teachers transcends the ability to carry out algorithms, but requires depth of detail needed for analyses of errors and explaining algorithm in learners language, representing the relationship between SKC and PCK. Wilson, Shulman and Richert (1987) define mathematical PCK as the knowledge that is used by a teacher to transform and represent knowledge of mathematics for teaching. This includes knowledge of approaches to topics, different ways of presenting mathematics, knowledge of children's strategies, common errors made and the knowledge of what material and media is available for

teaching. Kilpatrick (2006) believes that PCK is the phenomenon that extends beyond subject content knowledge and starts from a teacher's ability to prepare lessons right through to cultural influences that impact on learning.

Shulman (1986) believes that pedagogical content knowledge is a type of knowledge which surpasses both content and pedagogical knowledge. He describes it as the manner in which we address a subject in order to minimise learners' misconceptions and difficulties. It is the knowledge of what makes a topic easy or difficult and knowledge of the conceptions, preconceptions and misconceptions learners have of a topic or subject. In identifying learners' difficulties and misconceptions, Hadjidemitriou and Williams (2001) contend that teachers tend to underestimate the difficulty of tasks when they themselves share the misconception. Alternatively they overestimate the difficulty because questions may not require the sophisticated understanding of the concepts that teacher presume they need.

Shulman (1986) also highlights the fact that teachers need to be able to translate their perceptions of mathematics into classroom activities in order to provide their learners with the opportunity to understand and know. An (2004) believes that pedagogical content knowledge should be the central principal knowledge of teachers and stresses four aspects of PCK. These are the ability to (1) build on students' mathematical ideas and understandings, (2) address and correct students' misconceptions, (3) engage students in learning mathematics and (4) promote and support students' thinking about mathematics. Rasmussen and Marrongelle (2006) put forward the notion of a pedagogical content tool (PCT). This tool is a device such as a graph, diagram, equation, verbal statement, etc. that a teacher deliberately uses in order to connect to how learners think while teaching. It is implied that one would need knowledge of what kind of tool to use, and when and how to use it, in order to attain the specific teaching goals. They

suggest that this knowledge includes specific content knowledge, general pedagogic experience and knowledge of subject matter. They believe that PCTs which concentrate on what to do in classrooms are vital resources which teachers can use to think about a teaching activity in relation to envisaged learners' mathematical activity (Rasmussen and Marrongelle, 2006).

A study of the development of geometry PCK by Rossouw and Smith (1998) over two years revealed that students who attended the same in-service course ended up with vastly different levels of PCK at the end of the period of intervention. They concluded that "teacher eventually develop their own pedagogical content knowledge which is shaped by their own experiences and perceptions" (Rossouw & Smith, 1998:64). However, even though the PCK that is developed may differ from teacher to teacher, it is the quality of the PCK that they develop that is of importance - something teacher developers need to take into account (Shulman, 1986).

6. MISCONCEPTIONS IN MATHEMATICS

Swan (2001) highlights numerous reasons for mistakes in mathematics such as lapse in concentration, hasty reasoning, memory overload and failure to notice salient features in problems. Reasons not so easily explained include symptoms of deeper understanding and alternative interpretations of a situation. All over the world the same errors appear despite the differences in curriculum, pedagogical philosophy, teacher and language (Swan, 2001).

Askew and William (1995) suggest that teaching is more effective when misconceptions are identified and addressed by teachers and constantly making learners aware of these misconceptions. For instance, when we first learn multiplication and division we do so with whole numbers. The notion that multiplying is seen to imply that numbers get bigger and when dividing numbers get small. This thinking does not account for fractions and that in fact the exact opposite occurs when multiplying and dividing fractions. Hence to identify this

misconception that when we multiply, numbers get bigger and when we divide, numbers get smaller, clearly addresses the logical yet incorrect notion.

Swan (2001) asserts that misconceptions cannot be prevented, but skilled teachers can give explanations such that they are not encouraged or reinforced. This can be achieved by presenting problems that provoke confrontation with conceptual obstacles and discussing them in detail. Swan (2001) supports Askew and William's (1995) point that misconceptions should be welcomed in the classroom and made explicit to learners so that discussion about them can take place in order to produce more meaningful and longer lasting learning.

Olivier (1989) asserts that direct teaching to correct procedure does not eliminate the underlying causes for the erroneous behaviour. When learners are faced with an obstacle learners distort known schemas to overcome the hurdle. Olivier maintains that mistakes are made due to valid pre-existing knowledge and therefore if new ideas are to be built there need to be construction of new schema. Swan (2001) believes that misconceptions are not necessarily wrong thinking, but a concept in its initial stage of development or a generalisation of what is known by a learner and a natural phase in learning.

Léonard & Sackur-Grisvard's (1987) perspective on learners' misconceptions includes the following two precepts:

- Erroneous conceptions are so stable because they are not always incorrect. A conception that fails all the time cannot persist. It is there because there is a local consistency and a local efficiency in a limited area, that those incorrect conceptions have stability.

- For what problems are those conceptions mathematically correct? For what problems are they erroneous? It is only when we know the mathematical limits of the student's misconception that we will be able to know when their conceptions will fail, to prevent them, and eventually to teach them to students.

Léonard & Sackur-Grisvard's (1987 : 44 - 45)

Misconceptions are important indicators of what a student understands. Some misconceptions arise more often than others; these are developed in the early stages when one is unable to acquire appropriate meanings for symbolic statements (French, 2002). Misconceptions arise from interaction with everyday instances and the tendency to over-generalise correct understandings in the classroom (Hadjidemetriou & Williams, 2000). Swan (2001) suggests that certain rules learnt at an early age get generalised to later mathematics where they no longer hold true. A few causes of misconceptions are rules such as:

- When multiplying by ten just add 0 (this does not take into account decimals);
- The bigger number has the most digits (again decimals are ignored);
- Always take the smaller number from the bigger number (negative numbers are ignored); and
- Always divide the bigger number by the smaller (again negative numbers are ignored).

Certain misconceptions remain hidden and unexposed by teachers in the classroom, but misconceptions that are known are easier to address (Askew & Williams, 1995).

Misconceptions should not be avoided but used in teaching learners such that they are challenged by these misconceptions to revise and acquire mathematical concepts associated with graphs (Cornu, 1991). Williams and Ryan (2000) believe that the use of errors and misconceptions in teaching leads to effective learning as failure to review impulsive solutions to initial questions often leads to mistaken ideas. Swan (2001) believes that learners are not trained to re-evaluate their thinking, and that they should be lead to the view that their errors are simply common and broadly shared misconceptions, rather than careless working.

7. MISCONCEPTIONS IN GRAPHICACY

Van Dyke and White (2004) propose that learners struggle with graphs because they do not know which aspects of the graph to focus on or how to read graphical representations. They maintain that mathematicians naturally search for known features, i.e. intercepts, gradients, etc., while learners struggle to identify what to look for when working. Goerdt (2004) identifies some of these important features of graphs as rate of change, intercepts, zeros, asymptotes, local and global behaviour (maximum, minimum), and that learners need to take heed of these characteristics. Kerslake (1981) suggests that learners find it difficult to see that there are more points on a graph other than the points which they have plotted.

French (2002) states that algebra is often seen as senseless and not purposeful and believes that to counter this perception it should be made explicit that algebraic arguments have a three part structure i.e., representing elements in an algebraic form; transforming the symbolic expression (graphs); and most importantly interpreting the new produced forms. Coulombe and Berenson (2001) confirm what French says in that they believe algebra begins by studying symbolic representations to tabular and then graphical representations. What is missing is the student's ability to interpret these algebraic symbols and representations and research shows that

there are many common misconceptions related to functions and graphs. Some of these misconceptions are outlined below.

7.1. Slope-height confusion

Confusion of height versus slope arises when learners confuse the height of the graph with the rate of change (i.e. slope). For example in the graph, in Figure 2.2 one could ask "Which internet company costs more per hour at 2 hours?" The misconception would be evident when the respondent chooses call.com because it costs more at 2 hours, but does not recognise that the question is asking for slope, rather than height (Miranda, 2007).

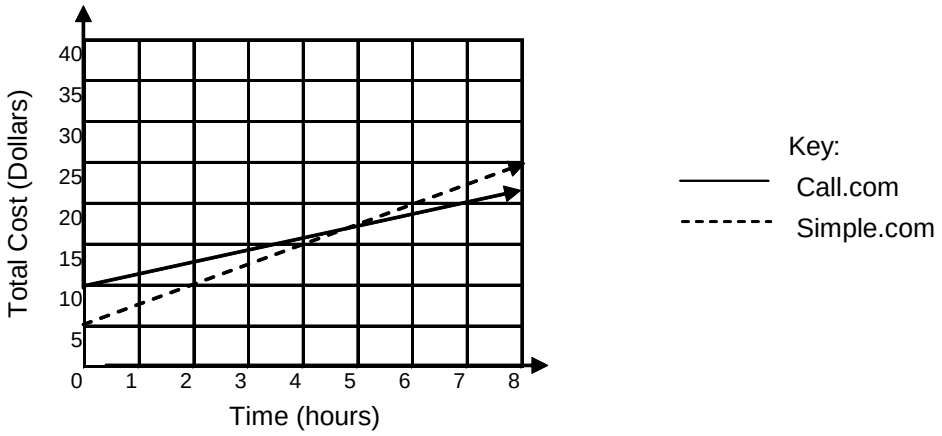


Figure 2.2: Total cost versus time for internet company costs. (Miranda, 2007).

Leinhardt, Zaslavsky and Stein (1990) suggest that this misconception arises due to the misinterpretation of confusing the two graphical features of highest value versus slope. Hadjidemetriou and Williams (2002) suggest that slope – height confusion may arise because pupils fail to understand that there can be a slope at a point. Thus even when learners distinguish

the correct slope over an interval, they may still struggle to construct a slope at a point because the notion of slope is not fully understood.

Hadjidemetriou and Williams (2000) in their assessment of graphical literacy discuss ‘increase versus value distraction’ while looking at a pointwise compared to a variational approach. The pointwise approach looks at corresponding points in the form of coordinates and the focus is on individual points. The variational approach looks at how something varies over time, where the emphasis is on how the values are changing and the steepness of the graph, i.e. gradient. Each approach is necessary but which one is engaged is dependent on the way in which a question is asked.

7.2. Inability to interpret graphs (graph as a picture)

Clement (1985) and Kerslake (1977) found that learners have an inability to see graphs as abstract representations of relationships but rather see graphs as literal pictures which are often in conflict with the correct interpretation and meaning of the graph. In a study by Kerslake (1981) secondary learners were given the following question:

Which of the graphs in Figure 2.3 represent journeys? Describe what happens in each case.

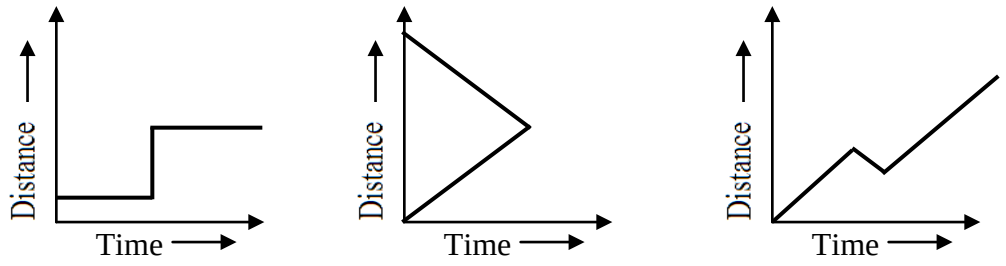


Figure 2.3: Distance time graphs (Kerslake, 1981)

Many saw the first graph as ‘going along, up and along’ or ‘climbing a vertical wall’ or ‘going east, then north, then east’. The second graph was interpreted as ‘going NE then NW’

whilst the final graph was often interpreted as ‘going uphill, then downhill, then up again’ or as ‘climbing a mountain’. Kerslake (1981) concludes that the spatial representation, appearance of the graph, distracts from the mathematical idea which is being communicated.

French (2002) refers to this misconception as the graph been viewed as a map of a journey, in which learners interpret the visual aspects incorrectly by seeing them as a literal picture rather than something abstract to interpret. Miranda (2007) suggests that learners often interpret graphs as pictures when they forget to apply algebraic relationships that they have learnt.

Confusing a graph and actual event is a very common misconception. The graph in Figure 2.4 indicates the speed of a vehicle with respect to time; learners do not understand that the vehicle could be going uphill when the graph of speed is declining (Bell & Janvier, 1981).

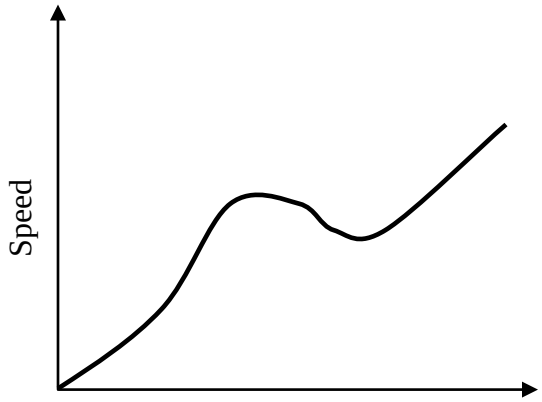


Figure 2.4: Speed time graph

Heid, Zbiek and Blume (2004) mention that learners sometimes see exponential functions as half a parabola; this observation comes from looking only at the shape rather than focusing on the important properties (gradient, intercepts, domain and range) of the graph. Janvier (1987) investigated teaching approaches to algebra and the different representations which includes symbolic, tabular and graphical representations. He suggests that using a problem-based

contextual approach where learners interpret graphs from familiar contexts creates a deeper appreciation for graphical construction and interpretation, data generating and other interpretative processes to take place. Janvier (1987) argues that in the traditional teaching approaches the interpretation step is absent as learners often work with the three representations and not interpretation processes.

7.3. Connection between the graph, equation and table

Algebra is language that is used to express the relationship between different quantities; it is also used as a tool to express and analyse the relationship between quantities that change (Van Dyke & Craine, 1997). Often learners fail to recognise the underlying equivalence between the graph, the equation of the graph, the verbal context or application and the table of values that the equation and graph represent.

Van Dyke and Craine (1997) argue that a person needs to understand the equivalence between the twelve directions shown below, in Figure 2.5, for translating the four representations as shown in the figure below.

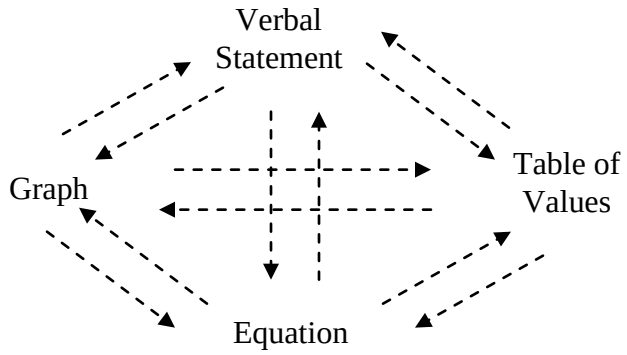


Figure 2.5: Van Dyke and Craines 12 directions of equivalence. (1997)

Flexibility to move between representations is an important mathematical development. The connectivity between multiple representations is often absent (Kuth, 2000). Kuth (2000) believes one must see that graphical and algebraic representations are ‘informationally

equivalent' (Figure 2.6), and use the relationship between equations and graphs as interchangeable. If the point satisfies the equation it lies on the graph and vice versa, if the point lies on the graph it satisfies the equation. When this equivalence is evident it demonstrates a deeper understanding of the relationship between graphs and equations.

Equation (Algebraic Representation)	$\rightleftarrows$	Graph (Graphic Representation)

Figure 2.6: Informationally Equivalent Graphical and Algebraic Representations.

7.4. Gradient / Rate of change

French (2002) defines the gradient as a crucial qualitative feature of any graph as it describes how the variables of a graph change. Rate of change is defined as the relationship between the independent variable and the dependent variable and this relationship is vital in developing students' understanding of rate of change (Coulombe & Berenson, 2001).

Crawford and Williams (2000) define slope (gradient) by the ratio of vertical change to horizontal change the traditional 'rise over run'. Crawford and Williams (2000) realise that learners can calculate slope easily enough with the given equation but seldom understand what it represents.

In their study Heid, Zbiek and Blume, (2004), maintain that because learners seldom encounter tables where the input values differ by anything other than 1, confusion arises and learners find it difficult to distinguish between amount of change (how one variable changes in relation to itself) versus rate of change (how one variable changes in relation to anther variable). Heid, Zbiek and Blume (2004) suggest that learners struggle to differentiate between change and

rate of change, and that teachers need to clearly distinguish between rate of change rather than the amount of change.

Learners often fall into the trap of linear interpolation where they attribute linear rate of change more frequently than appropriately without realising their error until asked for an explanation (Heid, et al., 2004). For example, students, who had just completed a section on exponential growth, were asked to estimate the population size at the end of week two given that the population size at the end of week one was 2 000 and at the end of week three was 8 000. The following interaction occurred (Heid, et al., 2004: 50):

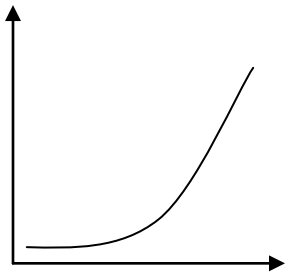
Student: *Five thousand! The population will be 5 000.*

Teacher: *How did you figure that?*

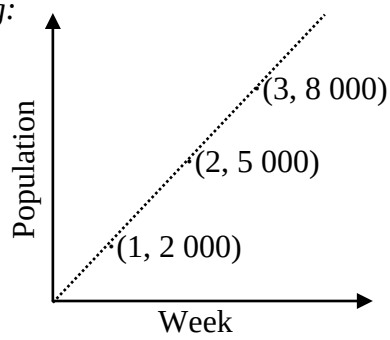
Student: *Well, 5 000 is halfway between 2 000 and 8 000.*

Teacher: *Can you draw a graph to show what you mean?*

Student's drawing:



Teacher: *Now show me a graph that illustrates the data you used. Student's second drawing:*



Teacher: *Are you satisfied with your two drawings?*

Student: *They look kind of different.*

Teacher: *Can you explain why?*

Student: *I'm confused: the curved graph says that it should be less than 5 000!*

This cognitive confusion arose because of the students' implicit linear model and the translation that every graph has a linear rate of change (Heid, et al., 2004).

7.5. Notation

Learners who are introduced to graphs are typically taught to see relationships with respect to the x and y, but when functional notation, f(x), is introduced to replace y, learners experience difficulties (Van Dyke & White, 2004). Learners do not understand what (x; f(x)) represents as they are used to the numerical representation of (1; -4). Lindquist (1989) found, in a study done with 11th graders, that evaluating a + 7 when a = 5, basic substitution, was 20 – 40% more successful than when the question was in the functional notation f(a) = a + 7, evaluate f(5), noting that f(a) is dependent on the changeable values of a. This could result due to algebra students understanding of what letters and variables represent. For example in the equation x + 5 = 8, the letter x does not represent a varying quantity but a fixed quantity of 3, because x + 5 = 8 is an equation and not a function. This blurred distinction of a root to an equation and a variable of a function create errors in thinking when trying to decipher what are functional and non-functional problems (Lindquist, 1989).

7.6. Language and misconceptions

Language can cause confusion which can lead to the formation of misconceptions. When meaning of words are different in mathematical contexts compared to their common usage (Dias,

2000). Hart (1981) suggests that language used in mathematics lessons is often technical and differs from learners' regular vocabulary and often need to be redefined. For example, Hart (1981), identifies the term 'straight' as a problem for certain younger learners, a line cannot be straight if it is slanting because straight is defined as being perpendicular to the edge of the page.

7.7. Scale

Surtrees (2005) identifies problems learners have with labelling of scale on graph axes. Firstly learners label the spaces and not the lines of a graph because learners' see the number of tracks and fail to see the axis as a number line.

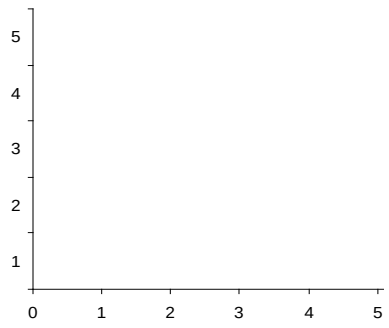


Figure 2.7: Number of tracks (y-axis) versus number line (x-axis)

Secondly, learners often label axes incorrectly by using an inconsistent scale usually dependent on the values which are supplied. Although axes are equally spaced, incorrect labelling creates a different representation of 1 unit.

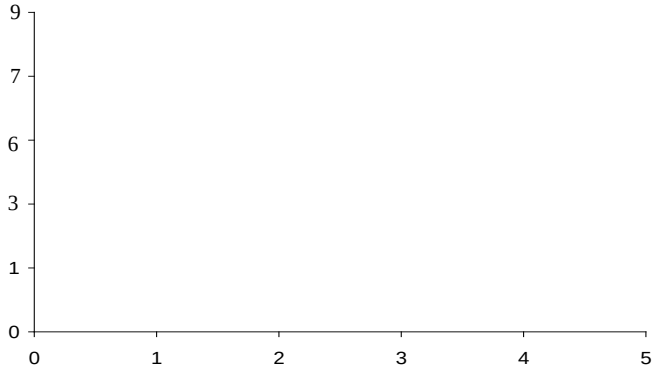


Figure 2.8: Incorrect labelling creating different representation of 1.

Thirdly, learners sometimes label the axes by starting at 1 rather than 0, thus indicating a confusion between counting, which starts at 1, and the fact that the axis is a scale of measurement.

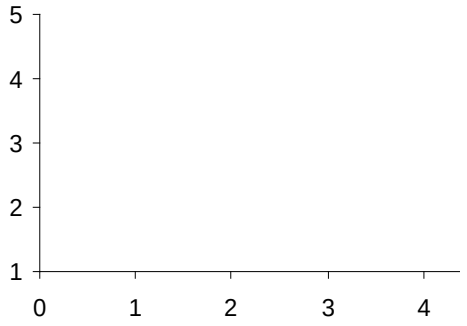


Figure 2.9: 0 and 1 confusion on axis as a scale of measurement.

Lastly, learners select inappropriate scale representations when using the same scale on both sets of axes, through the wrong assumption that it is required by convention. Recording the height of learners in 10 cm intervals on the y-axis, results in numbering of learners on the x-axis also to be 10 instead of 1 unit intervals.

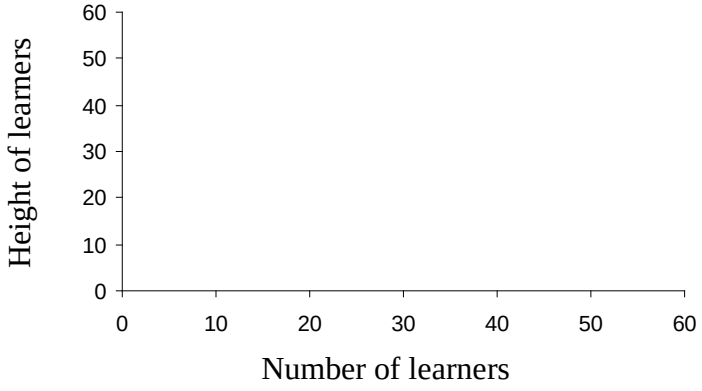


Figure 2.10: Inappropriate selection of scale due to ‘perceived convention’

7.8. Origin prototype

Hadjidemetriou and Williams (2002) define the origin prototype as the use of the origin as a prerequisite for any graph. No matter what the graph may be representing, the origin is used. Ryan and Williams (2007) suggest that when children are asked to draw graphs they will almost always draw them through the origin despite the context of what is being represented.

7.9. Contextual questions as a means of remedying misconceptions

Kerr and Maki (1979) define contextual questions as mathematical models. Mathematical models are the identification of a real life situation or problem that is simplified and modified, which means that do not truly reflect the real world. These models are then solved by using mathematical ideas. Kerr and Maki (1979) highlight that an important part of modelling is that the mathematical solution found is tested in comparison to the real world solution.

Contextual teaching according to Williams' (2007) view is the methodology which connects academic concepts to real world conditions which relate to everyday life. Students are thus able to make connections between mathematics and the real world and consequently see the importance of mathematics in life. Blinko (2004) suggests that the purpose for including contextual questions in tests and questionnaires is that many mathematical questions can be seen as complex and context actually clues the learner toward the appropriate solution. Another purpose Blinko (2004) outlines is that one does not just assess the skill and procedure of the mathematics but also the application thereof. Blinko (2004) believes that contexts can make abstract ideas more accessible to learners.

Julie et al. (1998) suggest that contextually based questions are effective only if the learner has understanding of the context. If the context is known, Julie et al. (1998) believe that a contextual question will aid learning and help counter misconceptions. However, context can negate the application of skills if one considers the language and layout of questions in context. These factors may intimidate the learner into feeling that they are inadequate to do the questions. Kerr and Maki (1979) believe that the usefulness of contextual questions or modelling depends on the appropriateness of the simplification of the real life context and the accuracy of the mathematical applications thereof.

8. MATHEMATICS EDUCATION IN THE SOUTH AFRICAN CONTEXT

With the institution of the first fully democratically elected government in 1994 a vigorous effort was mounted to reform South African schooling (Taylor, Diphofa, Waghmarae, Vinjevold, & Sedibe, 1999). The new government embraced the vision of the international progressive agenda for systematic change and established a number of key policy instruments in terms of constitutional rights, qualifications, school governance, school funding, language, teacher management and alignment of qualifications, curriculum, assessment and gender. Together these documents represented an impressively coherent vision for the fundamental transformation of the South African schooling system (Taylor, et al., 1999).

Possibly one of the most important reforms to take in terms of its immediate impact on the education of young adolescents was the introduction of a new curriculum. The curriculum builds on the vision and values of the Constitution of South Africa. These principles include social justice, a healthy environment, human rights and inclusivity, which reflect the principles and practices of social justice and respect for the environment and human rights, as defined in the Constitution. In particular, the curriculum attempts to be sensitive to issues of poverty,

inequality, race, gender, age, disability and such challenges as HIV/AIDS. The philosophy of outcomes-based education is the foundation of the curriculum and there are high expectations of what South African learners can achieve and aims at the development of a high level of knowledge and skills for all. Social justice requires that those sections of the population previously disempowered by the lack of knowledge and skills should now be empowered (Department of Education, 2002). Epistemologically the progressive consensus chosen is constructivist: learning must start in life experiences of students and classroom activities must be learner-centered and equip children for applying knowledge to real world problems. However, the implications of a constructivist approach for teaching and learning soon became one of the most controversial and difficult issues in South Africa education (Taylor, et al., 1999).

Morrow (2007) noted that while it is sometimes claimed that in post-1994 South Africa we have developed a bold and imaginative (a 'magnificent') set of education policies, admired across the world, the problem lies with lack of implementation. He asks the question 'Why do we have this problem? Who is to blame?' and his answer is firstly that South Africa still has some educational institutions (at all levels of the system) that remain stuck in apartheid traditions and mindsets and have not yet embraced 'transformation.' Secondly, and probably more importantly for this study, he noted that there are thousands of deficient schoolteachers, teachers who do not have the competences, or perhaps the willingness, to implement new policies capably. Educational change depends on what teachers do and think and Morrow (2007) believes that a high proportion of South African teachers have not yet accomplished the 'paradigm shift' they need to, if they are going to be competent implementers of fine policies.

In the past South African children have achieved very poorly on international tests such as the Third International Mathematics and Science Study (TIMSS) in 1996 and the Trends in

Mathematics and Science Study (TIMSS) in 2003. The South African Grade 8 learners in the study were the lowest scoring group of the 50 participating countries in both mathematics and science on both occasions (Reddy, 2006). The conclusion Reddy (2006) makes from the statistical analysis is that most learners are highly unlikely to achieve results that will enable them to enter engineering and scientific fields in tertiary institutions. She also notes that improved performance must take place in the lower levels of our schooling system if learners are to achieve satisfactorily at higher levels of schooling (Reddy, 2006).

Recommendations made by Taylor and Vinjevold (1999) from the President's Education Initiative (PEI) Research Project include improving teachers knowledge foundations in their subject content knowledge (SCK) in order to strengthen their conceptual understanding of the subject. They believe that improved SKC is vital to providing the necessary framework for developing their pedagogical content knowledge (PCK), the type of knowledge required before teachers are able to construct appropriate learning activities which meet the required competency levels of assessment standards of the new curriculum (Department of Education, 2002). They also point that, as the majority of teaching and learning in South Africa takes place in a second language, improving teachers' language proficiency in the language of instruction will result in more effective teaching and learning will taking place.

Taylor and Vinjevold (1999) concluded from the PEI research that teachers often do not engage learners deeply enough in terms of subject knowledge and skills. This results in learners operating at a cognitive level below what is required. They assert that this happens because teachers themselves not having sufficient SCK and PCK, in other words they feel that teachers do not understand their subject sufficiently to engage their learners substantively, resulting in dismal learner achievement. The PEI research revealed that the learners in this large-scale study

were seldom occupied with tasks dealing with anything other than elementary cognitive levels, thus never developing any higher order skills (Taylor and Vinjevold, 1999).

With respect to graphs and functions the new national curriculum statement (Department of Education, 2003) requires learners to investigate, analyse, describe and represent a wide range of functions and solve related problems (Learning Outcome 2: Functions and Algebra). This is in contrast to the requirements of the previous curriculum where learners were mainly required to sketch point for point, read off points of intersection where no emphasis was placed on interpreting global features such as general shape, intervals of rise and fall (Julie, Cooper, Daniels, Fray, Fortune, Kasana, et al., 1998). The current curriculum's assessment standards require, with respect to graphical characteristics, that in order to be judged as having met minimum standards the learners must be able to show evidence of being able to:

- Understand domain and range;
- Interpret intercepts with axes;
- Identify intervals by which the function increases/decreases;
- Understand the discrete or continuous nature of the graph;
- Investigate the average rate of change between two points of a curve and develop an intuitive understanding of the concept of gradient over different intervals;
- Demonstrate the ability to work with various types of functions;
- Recognise relationships between variables in terms of numerical, graphical, verbal, and symbolic representations, and convert flexibly between these representations (tables, graphs, words and formulae);

- Investigate, using mathematical modelling, real-life contexts describing a situation by interpreting graphs from a description of a situation, with special focus on trends and pertinent features (including health social, economic, cultural, political and environmental matters)

Covering the subject content in the curriculum is one of the many challenges that South African teachers face. One of the contributing factors towards not covering the syllabus is that teachers do not feel confident enough in their subject content knowledge to teach certain topics and therefore simply omit them (Adler, Slonimsky & Reed, 2002).

This study is an attempt to contribute to understandings of the problems faced by South African grade 11 teachers and second-language learners in a sample of rural schools in the Eastern Cape, South Africa, in the context described above. The study investigates the teacher's subject content knowledge, their PCK and their misconceptions about functions and graphs. A teacher questionnaire and interview protocol, which required them to predict how difficult their children would find items on a diagnostic test and to suggest likely errors and misconceptions that the children would make and to suggest methods/ideas they would use to help learners overcome these difficulties, was used to probe their levels of PCK. The study uses a diagnostic assessment instrument developed from literature on children's learning and misconceptions to investigate the graphical understandings and misconceptions of these the participating teachers and their learners.

9. SUMMARY

In this chapter different views of mathematical understanding are considered and what is actually meant by learning with understanding in mathematics is explored. How learners construct understanding and relate it to previously known concepts and procedures is also

described. Some of the definitions of subject and pedagogical content knowledge were aired, after which the relationship between these two types of knowledge were considered in terms of how they differ and complement each other. Misconceptions in mathematics as a field of research was discussed and some of the general misunderstandings and reasons as to why they occur in mathematics were illustrated. Misconceptions and problems directly relating to graphicacy were then explored in more detail before looking at the South African context with respect to the findings of the TIMMS study, the PEI research findings, and the South African curriculum for mathematics at school level. The transformation of the education system of South Africa is specifically discussed as are the changes to the curriculum with respect to graphicacy outcomes.

CHAPTER THREE

METHODOLOGY

1. INTRODUCTION

In this chapter I look at the research paradigms and research methodologies employed to study the answer to the question '*Do relationships exists between teacher's level of content knowledge, their PCK and their misconceptions about functions and graphs and their learner's achievement in terms of using graphs and, if there are, what is the nature of these relationships?*'

Different research paradigms are debated and the research design and methodology used in the study is outlined below. The setting of the research project is explained in greater detail. Insight is given into the data collection instruments and how and why they were used to collect data and how the data was analysed.

Reliability, generalisation and validity are terms debated within the mixed method approach to research and the ethical considerations taken into account are also discussed.

2. RESEARCH PARADIGMS

Guba (1990) defines the term paradigm as the basic set of beliefs that guides action. Johnson and Onwuegbuzie (2004) describe paradigm as the beliefs, values and assumptions which research communities have in common in terms of how they conduct research within a

research culture. Struwig and Stead (2001) describe a paradigm as a selection of mutually accepted modes of scientific practise, which directs research. Husén (1988) believes that the paradigm used determines how the researcher formulates the problem and how it is tackled methodologically.

Denzin and Lincoln (2000) outline four concepts of paradigms: axiology (ethics), epistemology (how does the researcher see the world), ontology (asking questions about the reality of the world) and methodology (how do we learn about the world). Lincoln and Guba (2000) look at issues confronting paradigms. One of these issues is accommodation and commensurability which focuses on how different paradigms slot into each other such that they can be practiced simultaneously in order to complement each other. They feel that certain paradigms may be incompatible, but still consider that within each paradigm certain mixed methodologies can be used and in some cases even advocated.

Burrell and Morgan (1979) identify four major paradigms based on continua from order to change, and subjectivity to objectivity. They represent these paradigms in four quadrants (figure 3.1).

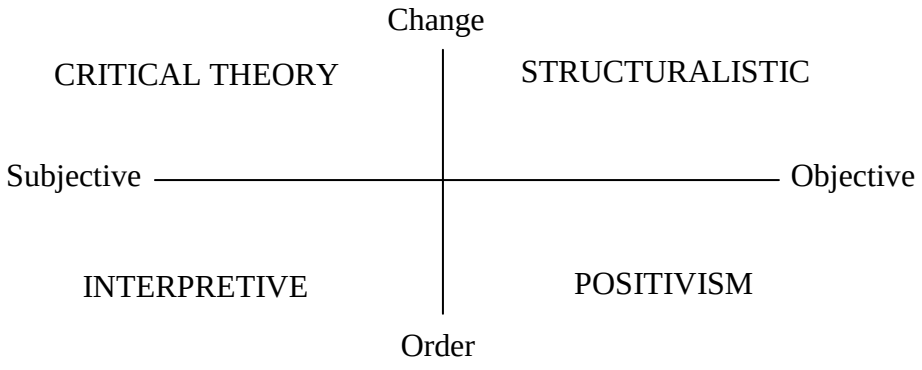


Figure 3.1: Research paradigms (Burrell and Morgan, 1979)

2.1 Positivistic Paradigm

Positivistic theory believes that research can be objective and value free, research makes use of scientific and quantitative research methods (McFarlane, 2000). McFarlane (2000). identifies the key indicator of positivists as the ability to predict outcomes. The interest is in identifying what factors determine the phenomenon and the study is successful when the factors coincide with general theory of society and lead to generalisable universal laws. Positivistic data are collected through experiments, surveys and secondary data and analysed numerically. Criticism of positivism is that research is seldom objective and value-free, and ignores the factors of social reality and the individuality of people (McFarlane, 2000).

2.2 Interpretivistic Paradigm

The interpretive paradigm examines how people make sense of their world. (McFarlane, 2000), talks about the process of becoming which he explains as people changing according to how they see themselves and where they wish to see themselves. Interpretive research realises that values are relative to social construct within individuals' situations, changing over time and cultural boundaries. Researchers are encouraged to understand these values at the time by utilising a variety of data collecting methods such as observation, participation and in-depth interviews.

Cohen, Manion and Morrison (2000) explain that interpretivistic theory is initiated with the individual and investigating how that individual interprets the world around them. It is from

this investigation that theory emerges rather than been predicted (Glaser & Strauss, 1967). Stringer (2004) describes understanding in the interpretive approach as being able to enter into the experience of another in understanding what they think and feel about certain events. Esterby-Smith, Thorpe, & Lowe (1994) suggest that interpretivistic research looks at defining the meaning people make of experiences by looking at what leads to these understandings of their world.

Fein and Hillcoat (1996) argue that behaviour is situation specific and too complex to generalise theories; the explanation of meaning pertains to a specific context. Although the theories developed from research in this paradigm cannot be generalised, Glaser and Strauss (1967) see research theories as a process for developing generalisations that can be tentatively applied in other situations and that research participants can relate the theory to their own experience. Fein and Hillcoat (1996) argue that general principles and themes from specific contexts could still be transferred to other settings, not through statistical methods but through stimulating thinking in application of theory to certain situations.

2.3 Critical Theory

Critical theory focus is on understanding social change. Popkewitz (1984:45) believes that the function of this theory is '*to understand the relations among value, interest and action and, to paraphrase Marx, to change the world, not describe it'*. Janse van Rensburg (1994) sees the researcher playing an active role in enabling individuals to learn about themselves and to enlighten them about the true reality of their circumstances. However, Lather (1991) feels that the discoveries made in critical research does not automatically lead to behavioural change.

2.4 Structuralistic Paradigm

This paradigm looks at a combination of objective and radical change and explains change with respect to fundamental conflicts due to political and economic crises (McFarlane, 2000). This paradigm is becoming less interesting to researcheres as more recent trends are moving toward a post-structural approach. Agger (1991) describes post-structuralism as a theory of knowledge and language which concentrates on the assumptions made in text which are analysed through deconstruction. Janse van Rensburg (1994) sees post-structuralism as the academic tools developed to understand present situations.

2.5 Mixing Paradigms

Guba and Lincoln (2005) realise that as time has passed that paradigms are starting to fuse such that, what was once seen as irreconcilable differences now seem to enlighten the other's argument. Subsequently one needs to investigate when paradigms reveal confluence and when they are inconsistent and discrepant. Guba and Lincoln (2005) believe it possible to "blend elements" of paradigms such that engaging research can be "the best of both worldviews" (p. 201).

Paradigms are largely categorised by the type of methodology employed by the researcher as a qualitative or a quantitative paradigm. Bryman (2006) advocates that there will always be clashes between objective and subjective paradigms when looking at the epistemological and ontological assumptions of these approaches. These clashes are between positivism and post-positivism (quantitative) versus interpretivist and phenomenological (qualitative) approaches.

3. RESEARCH APPROACHES

The debate as to whether one can combine research approaches, i.e. quantitative and qualitative data collection. Some researches believe that these methods should not be combined. Lewis-beck, Bryman & Liao (2004) suggest that quantitative and qualitative research methods are defined by the data collection and analytic methods used respectively and believe that qualitative content analysis becomes useful where statistical presentation becomes limited. Lewis-beck, et al., (2004) also suggest that a mixed method approach combining qualitative and quantitative data could increase validity of research through its flexibility to investigate both types of data.

The debate however is not limited to the data collection and analytical methods selected. Lewis-beck, et al., (2004) suggest that true mixed method approaches are impossible due to the contrasting epistemologies and ontological traditions. Denzin and Lincoln (2000) suggest that qualitative and quantitative paradigms are juxtaposed, in philosophical positions and methodologies. However, Guba and Lincoln (2005) suggest that positivistic and postpositivism paradigms are commensurable, hence the simultaneous practice of both is possible and that mixed method strategies may make common sense provided they share axiomatic elements. Guba and Lincoln (2005) caution, that commensurability can cause problems when researchers select obvious contradictory and exclusive axioms such as to suit their purpose.

There is a concern that quality of mixed methods design may be jeopardised when used incorrectly, as it appears to be in danger of becoming ‘fashionable’. On the other hand however, it is felt that mixed methods can deliver a superior quality research (Bryman, 2006). Bryman (2006) suggest three types of quality criteria in assessing mixed methods research, these include, convergent criteria (qualitative and quantitative data is subjected to the same criteria); separate criteria (qualitative and quantitative components are analysed according to different criteria) and

bespoke criteria (a new set of criteria for mixed method research). Each of these assessments may be suitable to different contexts dependent on the mixed method approach and the balance between the qualitative and quantitative methods used.

Sieber (1973) defines a mixed method approach as the examination of multiple processes of data collection by combining qualitative and quantitative approaches. Creswell (2003) mentions that all methods have their restrictions and that bias found in single methods may be neutralised by other methodology. Tashakkori and Teddlie (1998) believe that when one method is nested in another it enlightens different levels and or units of analysis that can be used in interpreting data.

A mixed approach is useful in that a researcher can collect complementary information but Johnson and Christensen (2004) caution that it is important to understand subjective and objective realities in the world. The fundamental ideal is to capture the different strengths of each method and know their different weaknesses. Cohen, et al., (2000) advocate that the selecting one method above the other is no longer necessary but rather using a combination so as to exploit the strengths of each.

Bryman (2006) suggests that looking from a technical level qualitative and quantitative research entails methods of data collection and data analysis rather than the deeper debates of a philosophical level. At a philosophical level these two research methods are unsuited and unable to be mixed. However, at a technical level they are frequently encouraged to be meaningfully integrated.

Creswell (2003) believes that a mixed method approach bases its knowledge on the practical view that the strategies employed to collect data, qualitative and quantitative

concurrently, is to best understand research problems dealing with both numeric and text information. This pragmatic approach sees the need for conducted research to develop deeper understandings of complex social phenomena, hence the rationalisation of the mixed method approach on a technical level while ignoring the epistemological and ontological discrepancies at a philosophical level (Bryman, 2006). Smith suggests boredom had risen for philosophical debates and urgency created for 'getting on with the task of doing … research' (Smith, 1996:162 – 163). Pragmatism enables researchers to overlook the academic disputes between qualitative and quantitative approaches and focus on the best methods employed to answer research questions. With the research question being the central factor which decides the approach methods (Bryman, 2006).

Johnson and Onwuegbuzie (2004) propose the acceptance of the mixed methods research as a third research paradigm and formally define it as '*the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study'* (Johnson & Onwuegbuzie, 2004:17). Fundamental to the method selection is designing a mixed method approach that effectively answers their research question.

Green, Caracelli and Graham, (1989) refer to triangulation of data in a mixed methods approach where the results from one method may inform data from the other. In constructing a mixed method approach Johnson and Onwuegbuzie (2004) believe that researchers need to decide if they are working predominantly in one paradigm or not and if the methodologies are conducted concurrently or sequentially. Figure 3.2 illustrates Johnson and Onwuegbuzie's (2004) mixed-method designs matirx showing nine potential different designs. Note that "qual" stands for qualitative, "quan" stands for quantitative. "+" stands for concurrent, "→" stands for

sequential, capital letters denote high priority or weight and lower case letters denote lower priority or weight.

Paradigm emphasis decision	Time Order Decision: Concurrent	Time Order Decision: Sequential
Dominant status	QUAL + QUAN	QUAL → QUAN QUAN → QUAL
Equal status	QUAL + quan QUAN + qual	QUAL → quan qual →QUAN QUAN → qual quan → QUAL

Figure 3.2 Mixed method design matrix

3.1 Quantitative Approach

Purists in the quantitative method observe that all data should be seen as physical phenomenon and treated as entities (Johnson & Onwuegbuzie, 2004). The quantitative paradigms see inquiry as being objective and value free is along with reliable and valid outcomes

(Nagel, 1986). Quantitative research looks at measuring and analysing the causal relationship between two variables (Denzin & Lincoln, 2000).

3.2 Qualitative Approach

Purists of the qualitative methodology reject positivism and believe in multiple-constructed realties and research is value bound (Johnson & Onwuegbuzie, 2004). Qualitative research is not dependent on experimentally examined or measured data with respect to quantity amount, intensity or frequency, but stresses the socially constructed nature of relationships between researcher and study object to answer the question how (Denzin & Lincoln, 2000).

3.3 This study's approach

The aim of this study is to answer the question '*Do relationships exists between teacher's level of content knowledge, their PCK and their misconceptions about functions and graphs and their learner's achievement in terms of using graphs and, if there are, what is the nature of these relationships?*'

To answer this question, the supporting sub-questions required a mixed method approach, where the emphasis was on qualitative data, but where quantitative data was used in a supportive role.

Quantitative research quantifies and attempts to clarify a relationship between two or more variables and is typical of a positivistic paradigm (McFarlane, 2000), however in this study these data were used to provide an initial framework in which both they and the qualitative data generated were interpreted, and were not considered from a positivistic perspective. Firstly, quantitative data were generated via diagnostic tests to provide an answer to the questions '*What*

is the level of content knowledge held by the teachers' and *'What is the level of content knowledge held by their learners?* The test used, which was based on those used by Hadjidemetriou and Williams (2002), measured both teacher and learner abilities and incorporated items that contained specific errors and misconceptions identified in literature (Hadjidemetriou & Williams, 2002). The test covered concepts indicated in the school curriculum, but which were not specifically taught before for the test in class or during the intervention.

Qualitative research explains and describes human behaviour, and is characteristically placed within the interpretivistic paradigm (McFarlane, 2000). Qualitative data were used to answer the questions '*What common misconceptions in terms of graphs are held by the teachers and their learners?'* and *'What is the level of pedagogical content knowledge held by the teachers?'* These questions require deeper and more rigorous probing as the fundamentals of PCK are wide ranging and therefore the open ended nature of the qualitative instruments (interview and questionnaire) allowed for better insights into the participating teachers' perceptions and thinking.

Johnson and Onwuegbuzie (2004) identified strengths in the mixed method research which support the design used in this study as the mixed- methods approach allows one to:

- Answer a broader and more complex range of research questions as one is not confined to one research method.
- Ensure that the strengths of one method can supersede the weaknesses of the other method.

- Corroborate the conclusions from evidence provided by both methods.
- Provide a more complete knowledge that can inform theory and practice.

4. RESEARCH DESIGN AND METHODS

As a starting point, the teachers completed a diagnostic test (appendix A) after which they immediately completed the teacher questionnaire (appendix B) designed to elicit their level of PCK. Questions on the questionnaire included predicting learner difficulty, identifying common errors and misconceptions, and ways that they could possibly rectify the problems learners encountered. The data generated by the test and questionnaire informed the design of the teacher interviews. These in-depth, open-ended, individual interviews allowed the interviewer to clarify and probe any ambiguities found between the tests and questionnaires, i.e. possible contradictions between teacher's subject and pedagogical content knowledge. The interviews were videotaped, transcribed and categorised according to common trends identified.

4.1 Questionnaires

Johnson and Christenson (2004) classify questionnaires as self-report data collections instruments which attempt to measure different kinds of characteristics. In this study two types of 'questionnaires' were used to measure subject content knowledge (SCK) and pedagogical content knowledge (PCK), i.e. the diagnostic test measured SCK while the teacher questionnaire measured PCK.

4.1.1. Diagnostic test

As noted earlier, the diagnostic assessment instrument was developed from literature on children's learning and misconceptions by Hadjidemetriou and Willams, (2001) which

investigated the graphical understanding and misconceptions of teachers and learners (Appendix A). This diagnostic questionnaire consisted of two parts; part one looked at different graphical ideas with in given contexts, while part two looked at graphing concepts using abstract mathematical type questions. The majority of the questions firstly required an answer followed by further explanation of the reasoning behind the given solutions. The latter part of these questions was probed by means of open-ended question (see appendix A). The diagnostic test was completed by both the teachers and their learners.

4.1.2. Teacher questionnaire

Schumacher and McMillan (1993) suggest that questionnaires are useful as they can be written for specific purposes. The purpose of the teacher questionnaire (appendix B) used in this study was to investigate teachers' perceptions of level of difficulty that their learners would experience when answering the diagnostic test items. The questionnaire used a five point scale ranging from very easy to very difficult. Secondly the questionnaire required teachers to suggest possible errors and misconceptions that their learners would experience with the diagnostic questions and then to suggest strategies they could employ to correct the identified misconceptions, as was the case in Hadjidetriou and Williams' 2001 and 2002 studies.

4.2 Teacher Interviews

Fontana and Frey (2000) contend that interviews are one of the most common and powerful ways in which we endeavour to understand each other. Open-ended interviews are designed to extract greater insight into the ambiguities encountered from quantitative data (in this case the test and parts of the teacher questionnaire data), and can be used to elucidate common misconceptions identified in literature and the pedagogical knowledge of the teachers.

In this study an individual, open-ended, in-depth teacher interview schedules were designed. The structure of each interview schedule was informed by the data generated via the diagnostic test and teacher's questionnaire responses. The qualitative nature of the interviews allowed the researcher to enter into the inner thinking of each of the interviewees and to gain an understanding of their perspectives (Patton, 1987).

Carr's (1996) 'interviews about instances' uses a set of stimuli to which the interviewees' were asked to respond. In this study the graphs and diagrams which the teachers were asked to interpret approximate Carr's (1996) 'instances'. Analysis of the interview questions addressed conflicting and interesting responses made in the diagnostic test and comments of the learners' misconceptions and their suggested intervention strategies to improve learner understanding.

The structure of the interview followed what Johnson and Christensen (2004) refer to as interview guided approach to explore specific topics and questions in order. Fontana and Frey (2000) believe that unstructured interviews allow for greater breadth due to the qualitative nature of the interviews, as one is trying to understand the complex behaviour of society's members. Each interview was videotaped so that the interviewer's attention could be focused on the interviewee and that valuable information was not lost.

5. DATA COLLECTION

Collecting the data took three stages. Stage one required the teachers to write the diagnostic test, which took place in a project contact session. Immediately after completing the diagnostic test they were asked to complete the teacher questionnaire

Stage two required the administration of the diagnostic test to the learners. The teachers took copies of the tests and learners wrote these under test conditions in the teachers schools (i.e., in a familiar context). Stage three included the development of the interview schedule and implementation of the interview process. After analysing the teachers' responses to both the diagnostic test and teacher questionnaire, individual in-depth interview schedules were drawn up for each teacher. Appointments were made with the teachers who were then interviewed individually either at their schools or before or after a project contact session. The interviews were video-taped. One of the teachers was not interviewed as he did not wish to participate in this aspect of the research process.

6. DATA ANALYSIS

The diagnostic test, questionnaire and interview data were analysed separately and then triangulated in order to provide indicators of the validity and reliability of the findings.

6.1 Diagnostic tests

The diagnostic test data generated by both the teachers and learners where analysed quantitatively. The tests were marked according to a memorandum which analysed the number of correct solutions to the items given. The results were first recorded separately for each group and in two different clustering's. The first cluster looked at part A of the diagnostic test which used the graphical test items within an everyday (familiar) context. The second cluster addressed part B which looked at non-contextual mathematically abstract graphical questions.

The teacher and learner responses were juxtaposed in terms of each teacher and their specific group of learners with respect to the mark (as a percentage) attained by the teacher in

comparison to the class average. Finally a collective overview of the teacher performance and learner performance was looked at by comparing the number of correct solutions as a percentage of all the teachers in comparison to that of all the learners in the study. Again these were split into part A (contextual) and B (non-contextual) of the diagnostic test.

6.2 Teacher questionnaires

Different sections of the teacher questionnaires were analysed either quantitatively or qualitatively according to the ability to accord a numeral or not. Quantitative analysis was employed to analyse the teachers' predictions of learner difficulty of each item as the teachers had to select a point on a 'degree of difficulty' scale (Appendix B). Each of the teachers' responses was compared to each other teacher in the group in terms of the contextual items and then the non-contextual questions.

The difficulty rating continuum, separated into five categories ranging from very easy to very difficult. If a teacher estimated that a question was very difficult then it was presumed that they expected between 0 – 20% of the learners to achieve the correct answer, while if they rated the item was as very easy, they expected 80 – 100% of their learners would successfully complete the item. The average of the teachers difficulty prediction was then compared to the actual learner achievement for each item from the learner completed diagnostic tests, once again in the separate clusters (contextual and non-contextual).

The teachers' responses to identification of learner errors and misconceptions and the suggestion of intervention strategies were analysed qualitatively. Each item was tabulated (see chapter 4) and comparisons and common threads were drawn and analysed in the discussion (chapter 5).

6.3 Interviews

Since the interviews were individualised and addressed different questions each was transcribed. From the transcriptions dialogue excerpts that supported common misconceptions identified in literature were identified and these extracts were then interpreted and interrogated.

7. SAMPLE AND SETTING

Keiskammahoek is a rural region in the Eastern Cape which previously fell under the former Ciskei which was a ‘self-governing homeland’ during the apartheid era. It lies in a basin at the confluence of the Keiskamma and Gxulu rivers below the Amatola Mountains. The name Keiskamma is of Khoekhoen (Khoi-San) origin, meaning either 'Puffadder River' or 'glittering water' (Keiskammahoek, 2008). The Anglican Church’s response to the needs of the Xhosa people suffering the impact of the wars of dispossession in 1854 resulted in the establishment of four Eastern Cape missions in the Keiskammahoek area: St Johns,St Lukes, St Marks and St Matthews. Extramural activities in these schools included a debating society and cricket. Education for boys was extended to secondary school level and in 1897 the Normal School for Girls was established. The proclamation of the gospel and the teaching of academic and industrial skills were promoted hand-in-hand. However, the apartheid state of 1948 put an end to the kind of education that helped shape the lives of so many of the learners in the Keiskammahoek region (Historic Schools Restoration Project, 2009).

The schools currently operating in Keiskammahoek are considered ‘previously disadvantaged’ due to the uneven support systems of the disparate educational systems that operated during this time and which were based on racial classification (Department of

Education, 2006) There is still limited access to educational resources and support from the current Department of Education in the Eastern Cape (Department of Education, 2006)

Five schools in the region were involved in the study. Five grade 10-12 mathematics teachers (one from each school), along with their grade 11 learners (n=66) in classes ranging from 9-20 learners, constituted the sample. The participants were selected because of their involvement in the Nelson Mandela Metropolitan University's Amathole project, an intervention that aimed at redressing the poor mathematical results of the area. The schools and teachers were offering Higher Grade mathematics for the first time ever as a result on the intervention.

8. RELIABILITY AND VALIDITY

Reliability is a measure of consistency, i.e. would the same results be found if the study was replicated at another time and date with a similar group (Cohen, et al., 2000). Smith (2006) discusses Campbell and Stanley's (1963) and Cook and Campbell's (1979) four types of classical typology of validity – statistical, construct, internal and external which each relate to four types of inferences within a positivistic paradigm. Smith argues that this kind of validity is incongruous to qualitative approaches as they do not satisfy the same technical requirements and procedures. Instead the criteria should pertain to words such as descriptive adequacy, fidelity, comprehensiveness, authenticity and ecological validity. Authenticity is the key to determining that the inferences made during the data analysis are as correct as possible.

Duit, Treagust and Mansfield (1996) express the opinion that diagnostic test items have t limitations in that it is difficult to interpret learner responses, especially if the items have not been field tested. However, Cohen, et al. (2000) point out the advantages of using published tests as they are objective, piloted and refined, standardised, reliable and valid. In this study the

validity and reliability of the diagnostic test and questionnaire data hinges on the reliability and validity of Hadjidemetriou & Williams (2001) diagnostic test. In turn, their study informed the design of the questionnaire, and in this way an attempt was made to maintain the adequacy, fidelity, comprehensiveness, authenticity and integrity of the data collection instruments.

At this stage it should also be noted that the nature of pedagogical content knowledge is in itself very difficult to assess. By definition pedagogical content knowledge is knowledge that surpasses content or subject knowledge and is knowledge that minimises learners' misconceptions and difficulties (Shulman, 1986). Inferences from the questionnaires and interviews were therefore used to further assess, investigate and interrogate the level of content and pedagogical content knowledge held by the participating teachers.

9. ETHICAL CONSIDERATIONS

Before starting the project it was made clear to the participants that various research undertakings would be conducted throughout the duration of the project. The teachers and learners of the project were verbally informed of the purpose and procedures of the research that would take place and they voluntarily agreed to take part in the proposed research. Dienar and Crandall (1978) define informed consent as the procedures in which individuals choose whether to participate in an investigation after being informed of facts that would be likely to influence their decisions.

The participants where assured of anonymity and confidentiality and were to discontinue their participation in the project at any stage. They were also told that the findings of the research would be made available to them on completion of the intervention.

Cohen, et al., (2000) reminds researchers that in their search for truth they have a responsibility to take cognisance of the effects that their research may have on its participants. As this study took place within a project where confidences and experiences were shared over a three year period, open relationships of trust and mutual respect were developed over time.

10. SUMMARY

In this chapter research paradigms are discussed and those employed in the study are looked at in greater depth. The research design and methodology is outlined and the dilemmas inherent in using mixed methods are highlighted. The data collection instruments, data collection techniques, and methods of analysis are outlined. The setting and sample used are described and the ethical issues taken into consideration are discussed. The question of reliability and validity are considered and the attempts made to ensure that the research was done within acceptable ethical standards are also considered.

CHAPTER FOUR

RESULTS

1. INTRODUCTION

In this chapter the quantitative results of the learner and teacher diagnostic tests are considered and whether there is any teacher-learner correlation is investigated. The qualitative responses to the teacher questionnaires are reported and excerpts from the teacher interviews, diagnostic tests and questionnaires are examined. Graphical misconceptions encountered in the learner and teacher responses in the diagnostic test are highlighted, as are misconceptions that were aired during the interview discussions. Any overlaps between the two group's misconceptions are also noted.

2. QUANTITATIVE ANALYSIS

The teacher and learner diagnostic tests and the teacher questionnaires generated the quantitative data that are presented below.

2.1 Learners' Diagnostic Tests

A sample of 66 learners completed the diagnostic test. The first six questions were contextually based while the questions in section B were not set within any context. The learners' results for these two sections of the test are shown in figures 4.1 and 4.2. The overall

average for the diagnostic test was just over 20% and most of the correct solutions for the individual items had very low mean scores, the exception being the transport question in section A (contextually situated questions) where the averages exceeded 40% (see figure 4.1).

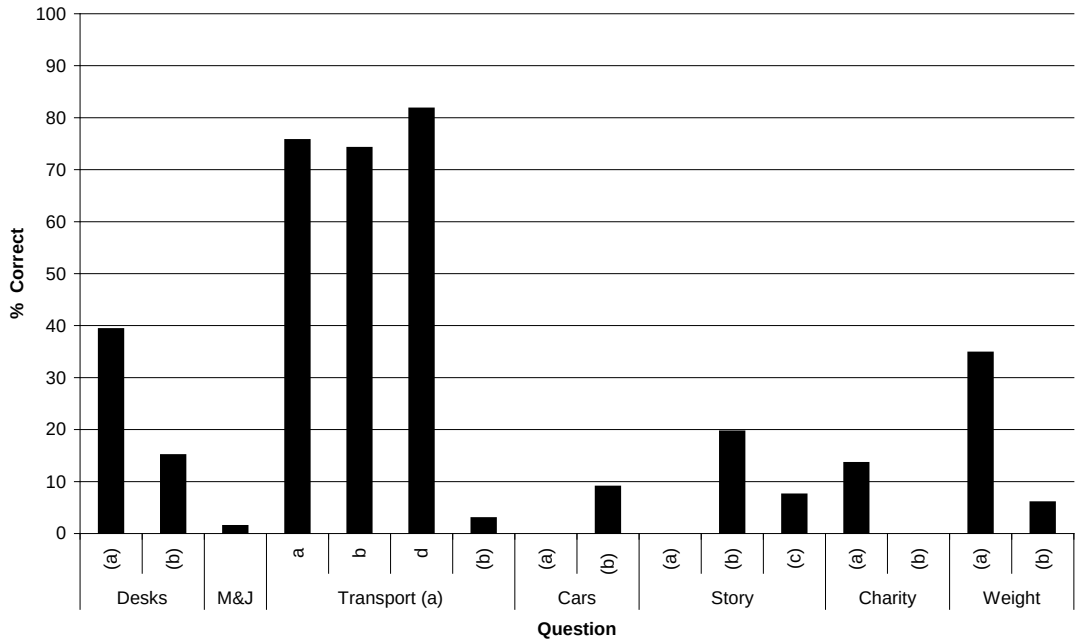


Figure 4.1: Learners Response to the contextual questions in the diagnostic test.

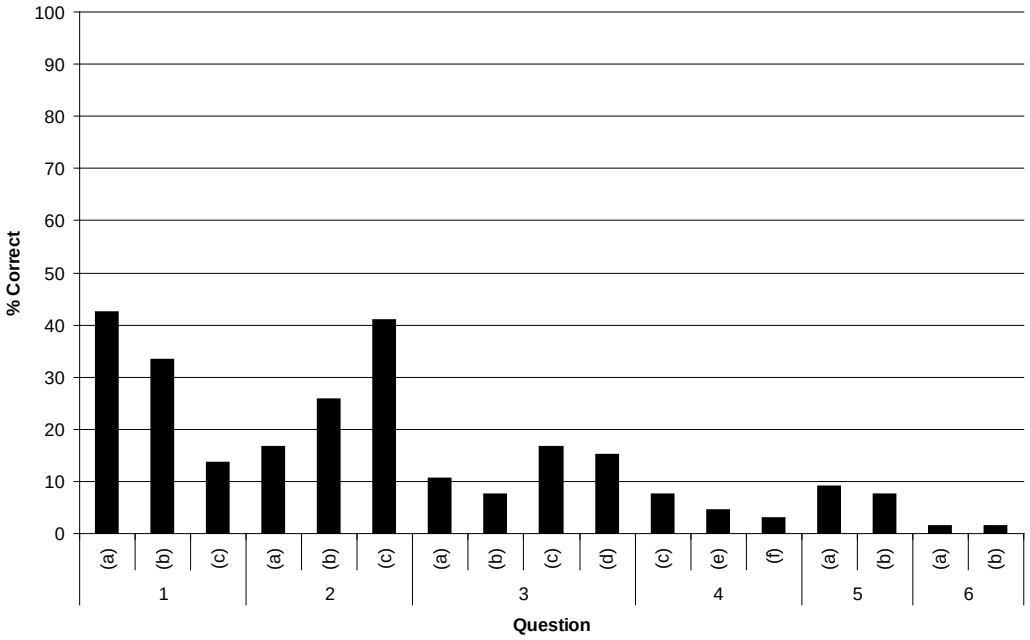


Figure 4.2: Learners Response to the non-contextual questions in the diagnostic test.

2.2 Teachers' Diagnostic Tests

The five participating teachers completed the same diagnostic test as the learners, scoring an average of 58% for the test. Their results, for the contextual questions in part A and are shown in figure 4.3 and the non-contextual results are shown in figure 4.4. The participants scored higher in the non-contextual questions in comparison to the contextual questions. Only four questions in the test were answered correctly by all the teachers, namely 1(c), 3 (c) and (d) and 5(a), all of which were in section B, the non-contextual part of the test. There were three questions in the test that all the teachers failed to give the correct solution, i.e. transport (b) and weight (b) in the contextual sections, as well as 6(b) in the non contextual set of questions.

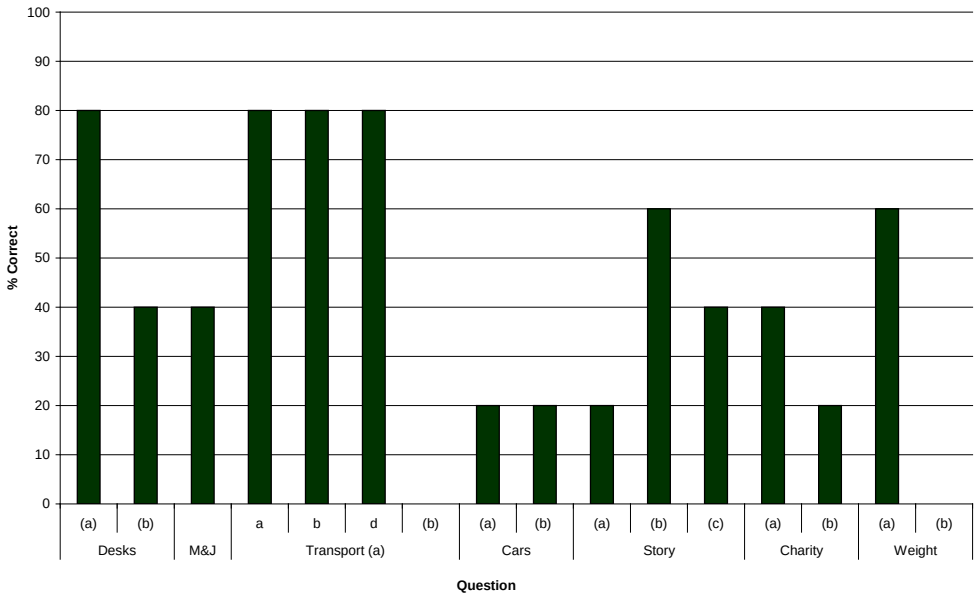


Figure 4.3: Teachers Response to the contextual questions of the diagnostic test.

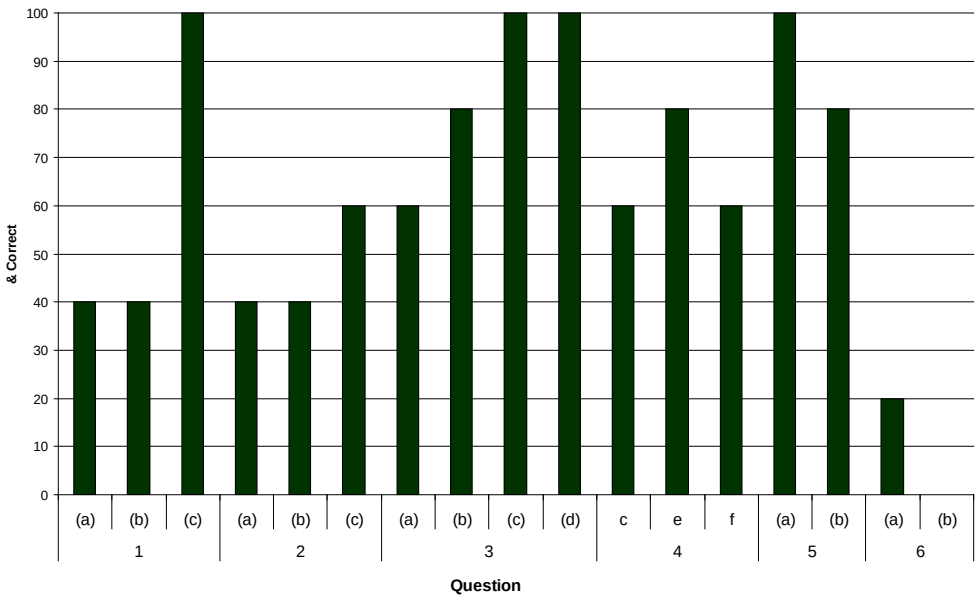


Figure 4.4: Teachers Response to the non-contextual questions diagnostic test.

2.3 Teacher-Learner Comparison

As noted earlier, initially it was thought that the teacher-learner data could be treated statistically in an attempt to determine whether any differences were statistically significant or not. Initial attempts revealed that the small sample size of teachers rendered this approach not viable. As such, the data are presented graphically for visual inspection. The first association shown in figure 4.5 is the link between the average scores as a percentage that the teachers achieved against the average class achievement of his/her learners in the diagnostic test.

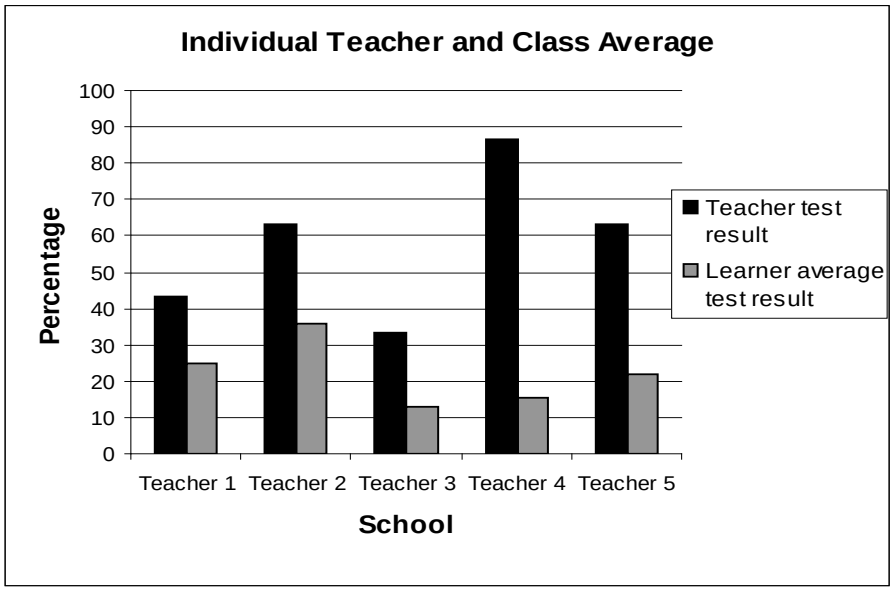


Figure 4.5: Teacher-learner correlation according to achievement in diagnostic test.

A comparison of learner and teacher achievement in the diagnostic test (as percentage) is presented in figure 4.6 and in figure 4.7.

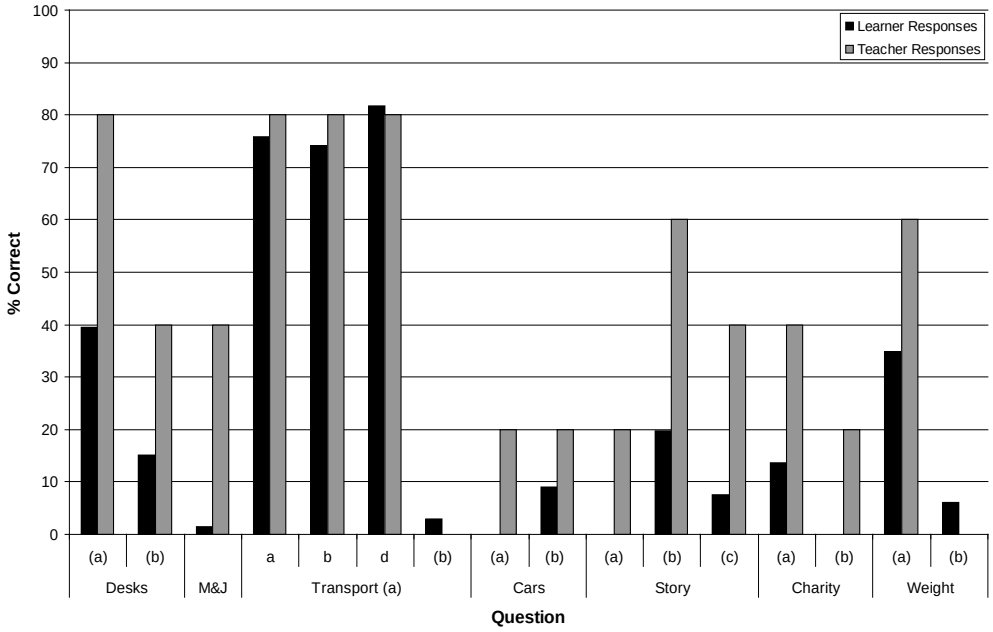


Figure 4.6: Comparison between teacher and learner responses to the contextual questions in the diagnostic test.

The contextual question on transport shows the closest match between the teachers and their learners' responses. For the rest of the questions the majority are better answered by the teachers with only three cases where the learners did marginally better than their teachers (see transport (a) number d and (b) as well as weight (b) (see figure 4.6)

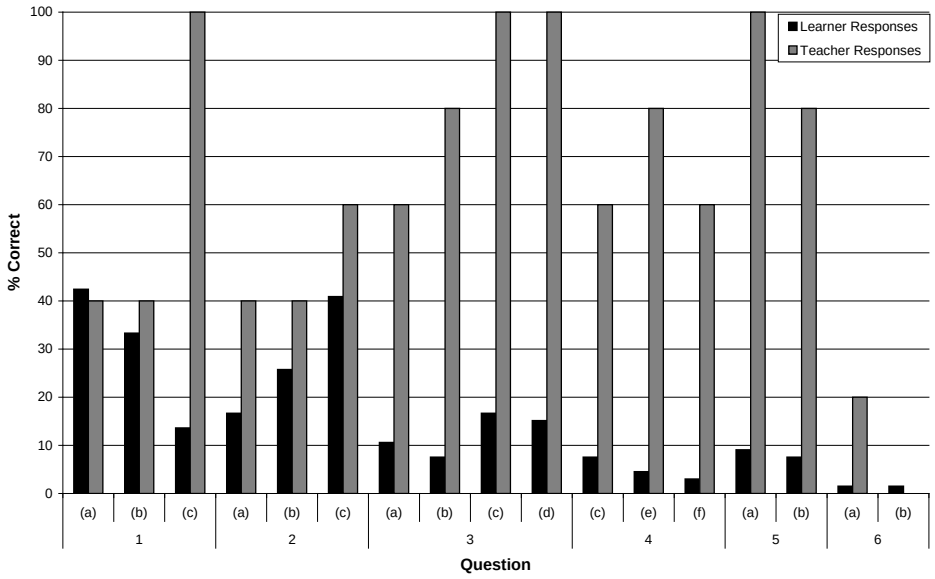


Figure 4.7: Comparison between teacher and learner responses to the non-contextual questions in the diagnostic test.

The teachers did substantially better than their learners' in the non-contextual section of the test, especially on questions three, four and five. Question 1 (a) and (b) showed the closest match between teacher and learner achievement. In questions 1 (a) and 6 (b) the learners performed fractionally better than their teachers did.

2.4 Teacher Questionnaires

The first requirement of the teacher questionnaire looked at the participating teachers' formal qualifications and experience (number of years that they had been teaching mathematics).

These data are recorded in the table 4.1. Each teacher number correlates with the teachers numbered in figure 4.5.

Table 4.1: Qualification and experience of the teachers participating in the intervention and the number of learners in their class

Teacher	Qualification	Number of years teaching mathematics	Number of learners
1	HDE (Higher Diploma in Education)	9	9
2	STD (Secondary Teaching Diploma)	8	9
3	SPTD (Senior Primary Teacher's Diploma)	13	9
4	STD (Secondary Teaching Diploma) FDE (Further Diploma in Education)	10	19
5	SSTC (Senior School Teaching Certificate)	29	20

Secondly the teacher questionnaire, which was designed to acquire insight into the PCK of teachers, asked the respondents to rate the level of difficulty that they thought their learners' would experience for each item in the diagnostic test. The difficulty ratings fell on the following five point scale: Very Easy, Easy, Average, Difficult and Very Difficult. Figure 4.8 shows the teacher perceptions on the difficulties of each contextual question in the diagnostic test while figure 4.9 looks at the non-contextual questions.

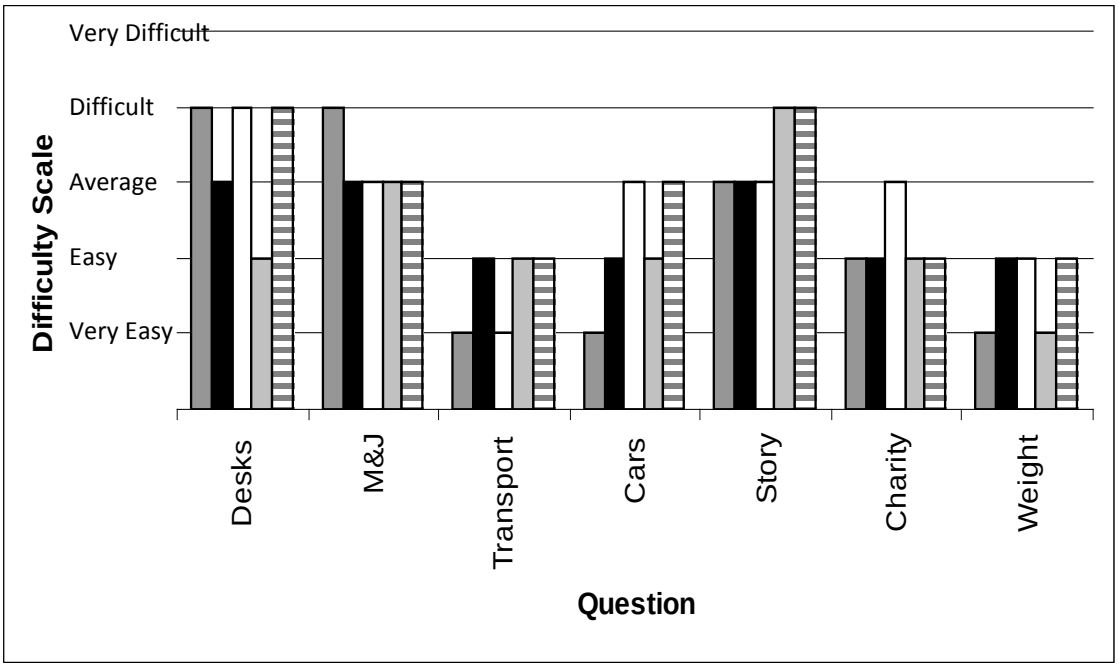


Figure 4.8: Teacher perception of the level of difficulty of the contextual questions in the diagnostic test (each bar represents one of the 5 teachers responses to each question)

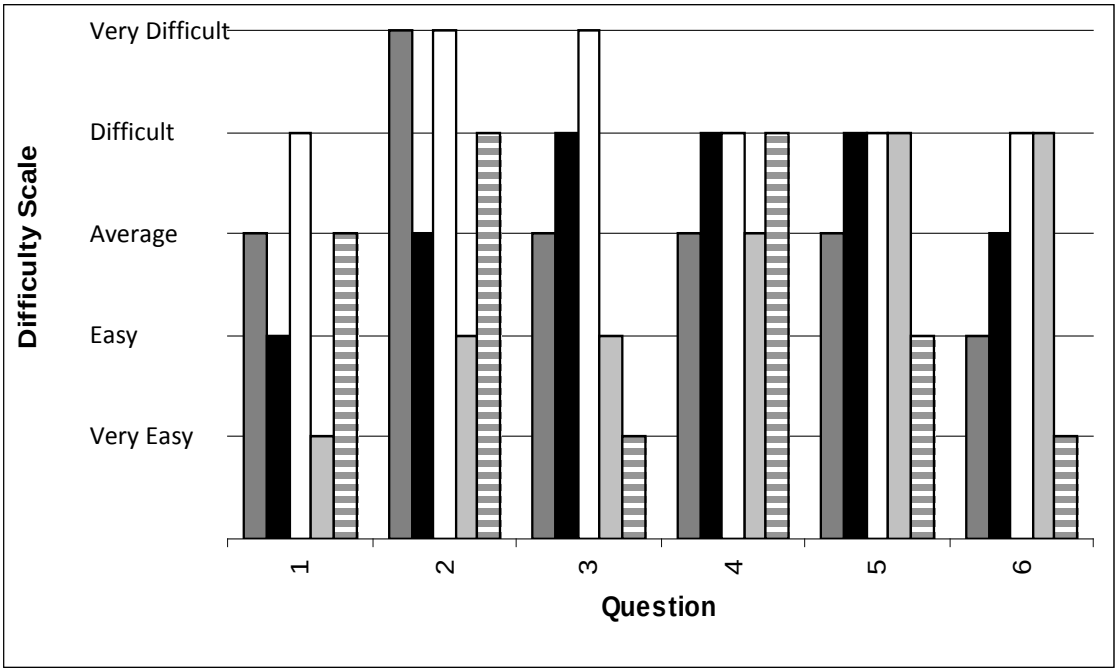


Figure 4.9: Teacher perception of the level of difficulty of the non-contextual questions in the diagnostic test.

The difficulty scale was converted to an estimation of how many learners they felt should attain the correct answers, by rating very easy as 80 – 100% correct answers through to very difficult as 0 – 20% of the learners achieving the correct answers. The estimations of the teacher predictions of correct answers were compared to the actual percentage of correct answers achieved by the learners with respect to the contextual questions (figure 4.10) and the non-contextual questions (figure 4.11).

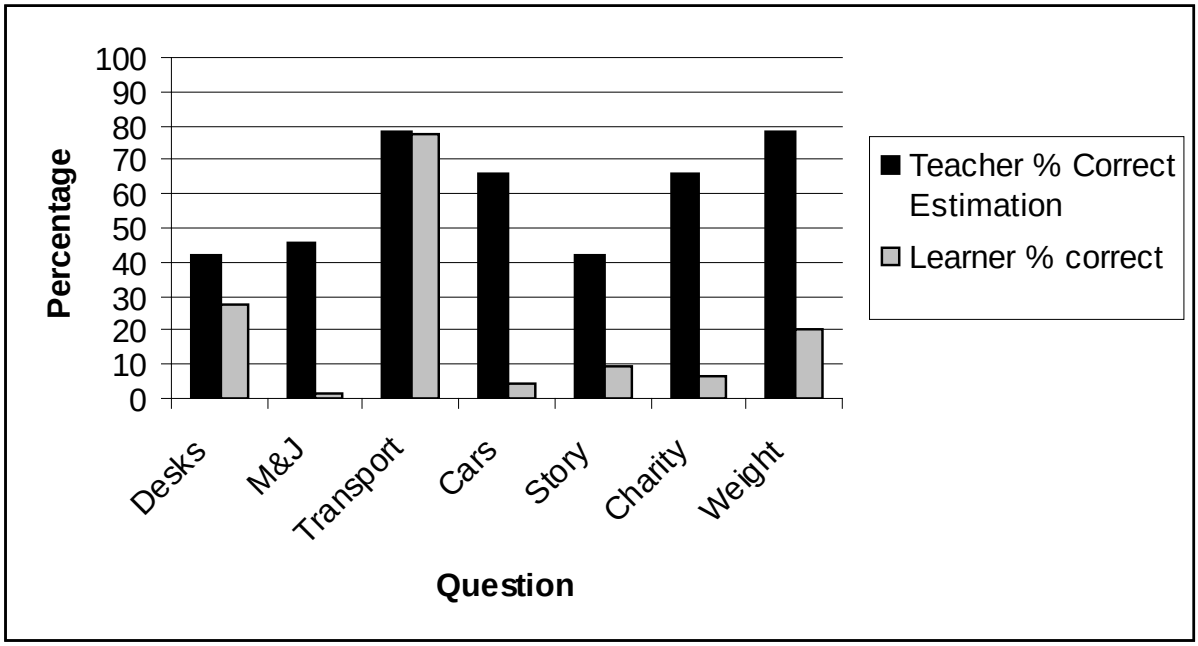


Figure 4.10: Comparison of teacher expectations with learner achievement in the contextual questions.

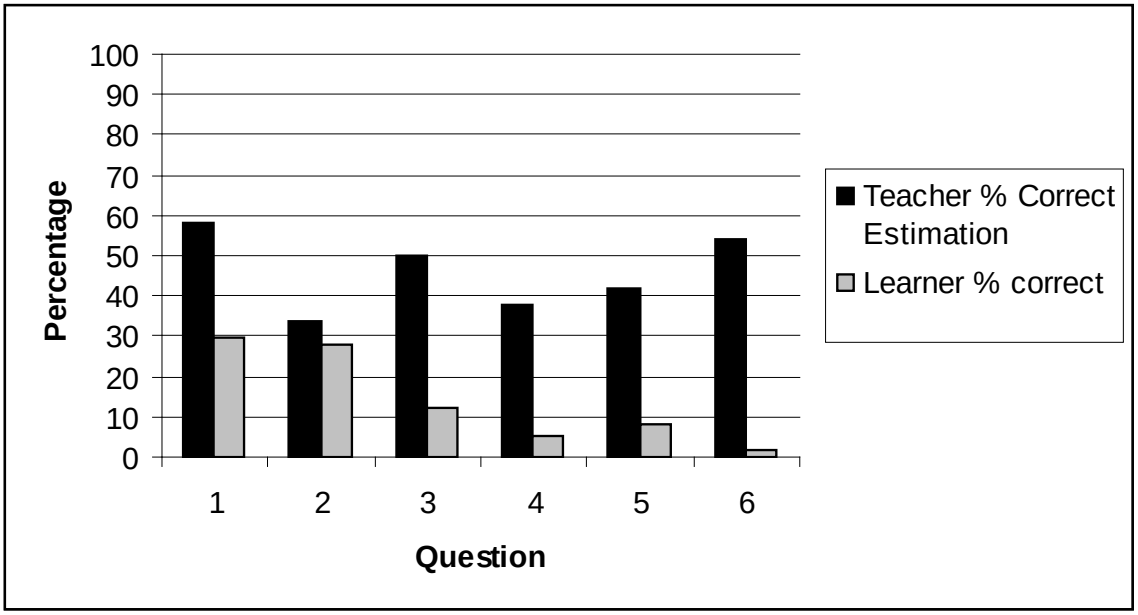


Figure 4.11: Comparison of teacher expectations with learner achievement in the non-contextual questions.

The teachers overestimated the difficulties that their learners would experience with all the items on the diagnostic test. The item on transport in part A (contextual questions) showed the best prediction of difficulty by the teachers. The rest of questions showed learners underachieving in respect to their teacher's estimation of how difficult their learners would find the questions, i.e. they expected that they would do better than they actually did.

3. QUALITATIVE ANALYSIS

Qualitative data were generated via the teacher questionnaires and the interviews held with the participating teachers.

3.1 Teacher Questionnaires

Along with qualifications and teachers rating the difficulty of questions, as discussed above in 2.4, the teacher questionnaires also asked the teachers to identify misconceptions and difficulties that their learners would experience in terms of the questions asked in the diagnostic test. Further, they were asked to give examples of how they would address these in their teaching. Suggestions made by the teachers are recorded in table 4.2. Omissions in the table reflect what the teachers chose to omit in the questionnaire.

Table 4.2: Learner difficulties and misconceptions as identified by the teachers

Question	Teacher	Learner Errors & Misconceptions
Desks	1	Confusion in identifying the Brian's coordinates. Which desks add up to 4.
	2	To spot Brian. Thinking of the position of learner B.
	3	They will have a problem with the axes – which is x and which is y.
	4	Axes, they are used to starting at the origin, having positive x-values on the right-hand-side of the x-axis.
	5	Forget that rows are along the path of the teacher as he/she enters the class and columns as he/she moves between the desks.
Mary & John	1	Language will be a problem before attempting the problem.
	2	To think about the slope of when they are running.
	3	In plotting the graph.
	4	Who runs first and who start by walking.
	5	They will confuse that when you walk the slope it is not so steep like as you are running the slope is steeper.
Transport	1	Problem will be in reading the graph of which one will take a longer distance.
	2	Drawing a graph.
	3	Approximating times of the different types of transport.
	4	Speed, they must have an idea of how fast the mode of transport is compared to the others.
	5	Forget that as you move by car the slope is steeper than as you ride bicycle or walk.
Cars	1	Interpretation of graph.
	2	To spot the starting distance of Car B.
	3	Confusion about which is the faster and which is the slower of the two.
	4	Speed and point of intersection of two cars.
	5	They forget that the distances are not the same from home of car A and B.
Story	1	Interpretation of graph. Reading of time and distance since it is illustrated by a curve.
	2	Their problem is to know when the graph is decreasing or increasing.
	3	Approximating time and distance travelled.
	4	Reading the graph correctly.
	5	Learners are not all good listeners to stories.
Charity	1	This one is clear, since they are used to fundraising concerts.
	2	To draw the graph.
	3	In plotting their own graph.
	5	They will have little difficulties as they are involved in concerts.
Weight	1	This is what they experience in their daily lives that girls grow faster than boys. I don't think they will have a problem.
	4	Reading graph correctly.

	5	They will have some difficulties as they think boys grow faster than girls in weight.

1	1	Finding the coordinates. Calculating slope.
	2	Think that the graph is not increasing constantly.
	3	It will be difficult to get used to the pattern used in drawing the graph.
	4	Multiplying correctly.
	5	Forget that one adds the same number to itself when finding multiples.
2	1	Interpreting and understanding of squaring of numbers.
	2	Square roots and squaring numbers.
	3	Have a problem with the blocks on the graph.
	4	Square correctly.
	5	Have a tendency to add terms when raising powers e.g: $3^2 = 3 \times 2$
3	1	Interpretation of the meaning of the different curves in this graph
	2	Interpreting graph C.
	3	Have confusion with these values of y and x.
	4	Difficulty in analysing the graph.
4	1	Terminology. Eg: meaning of constant.
	2	Forgetting that graph F is increasing constantly.
	4	Analysing the term changing at a constant rate.
5	1	Understanding of parallel lines and the influence of slope in a graph.
	2	Forgetting that A and B are parallel – their slope is equal.
	3	Have a problem with the axes that are not labelled with numbers for guidance.
	4	Looking at the greatest slope.
	5	A slope is just like where you walk up a hill.
6	1	Interpretation of the question. Understanding shaded area.
	2	Forgetting rate as how fast is the change of the two graphs.
	4	Not looking at the shaded area only.

Table 4.3: How teachers would address difficulties and misconceptions identified in table 4.1.

Question	Teacher	How would you correct the errors and misconceptions?
Desks	2	Let them play chess/drafts so that they can find their positions.
	3	Let them identify which of number stands for x and y in the coordinates.
	4	Put desks in the order they are familiar with.
	5	Move between desks, showing each row and column means a point where they meet.
Mary & John	1	They must read the question first and try and understand it before answering it.
	2	Let them practice how to plot a graph and how to find a slope of many lines.
	3	Help them; show them how to plot the graph.
	4	Explain clearly who runs for a certain period of time and walks for the other.
	5	Mary first moved fast by running and then walk.First fast then slower.
Transport	1	Consider which one is faster. Then the faster one will take less time than the slower one.
	2	Let them take the situation to their daily lives and show them how to plot graphs.
	3	Analyse how the transports travel as the are all at their disposal, they use bicycles, buses, cars and some walk on foot to school.
	4	It seems as if a person riding a bicycle has reached school after 20 minutes. Explain clearly what is needed in graph 2.
	5	Look at how fast each of the transport goes.
Cars	1	Understand the speed of each and the distance the cars have taken.
	2	Show them the meaning of the two graphs first. (draws two graphs, one through the origin and one with a positive y-intercept)
	3	Help guide them towards realising what's happening in the two cars.
	4	Explain/Show them how the graph is where the car has high speed.
	5	That cars at the same rate on the speedometer are at the same speed if of the same size.
Story	2	Let them draw so many graphs and find their slopes first.
	3	I will make practical examples using them to travel the tuck-shop.
	4	They must be able to compare two quantities.
	5	Try and analyse when listening to the story.
Charity	1	They must relate this to their fundraising activities so as to be able to understand this question.
	2	Teach them how to draw graphs of proportion and what it is.
	4	When you have nothing you always start at the origin.
	5	They will have few problems.
Weight	1	This is what is experienced by them that boys are growing slower than girls at the age of 14.
	2	Let them do problems involving "word problems" and how to analyse the situation and teach them how to interpret graphs.
	4	Teach them carefully how to read the graph.
	5	They do not know when you gain weight you grow. Show them that it means growth.

1	1	Take any points of the line and then calculate the gradient using the formula they know.
	2	Show them how the graph increases first.
	3	Assist my learners in identify the scale used to draw the graph.
	4	Multiples to be taught thoroughly.
2	1	Square the given coordinates.
	2	Let them learn and memorise the types of slopes and their meaning.
	3	Help them by going through the easy method of calculating the slope.
	4	Squaring numbers and finding square roots to be revised.
	5	Give them many examples of powers and that 2^2 means two factors of 2. Eg: $2 \times 2 = 2^2$
3	1	Try to understand the graphs by looking at the shape of the curve whether it is decreasing or increasing.
	2	Show them when the graph is increasing and decreasing.
	3	Give meaning to increase and decrease and the direction to which x and y values increase and decrease.
	4	Reading the graph carefully, knowing when the variable is increases/decreases is important.
4	1	Understand the terminology correctly.
	2	Give them the graphs with given variables and numbers and let them calculate slopes.
	3	I will label the axes and write in values along the x and y-axis.
	4	Graph of the rate of changed to be revised.
5	1	Understand what parallel means and the influence on the gradient in a graph.
	2	Let them practice graphs very carefully.
	3	Make my learners aware about change of graphs along the x and y-axis.
	4	Teaching gradients carefully and showing them when it is the greatest or least will help here.
6	1	Read instructions carefully.
	2	Give them many graphs to practice how to find slopes.
	4	Pointing out to them to concentrate on the shaded region.

3.2 Teacher Interviews

The teacher were interviewed in an attempt to seek clarity on some of the ambiguities detected between the diagnostic test and the questionnaire responses, and focused on specific misconceptions that became apparent after analysing the tests and questionnaires. Excerpts of these interviews are presented where appropriate.

4. GRAPHICACY MISCONCEPTIONS

There appeared to be evidence of misconceptions held by the teachers and learners in the diagnostic test data. During the interviews with the teachers these issues were discussed in depth and, where applicable, the teachers' ideas as to how they could be addressed have been included in the reporting below.

4.1 Slope-height confusion

The weight question in the diagnostic test was developed to probe the misconception of slope-height confusion (figure 4.12). The question consisted of two parts. Question (a) asked what the difference in weight between boys and girls was at the age of 14, while (b) asked which group is growing faster at the age of 14 and to explain their answer.

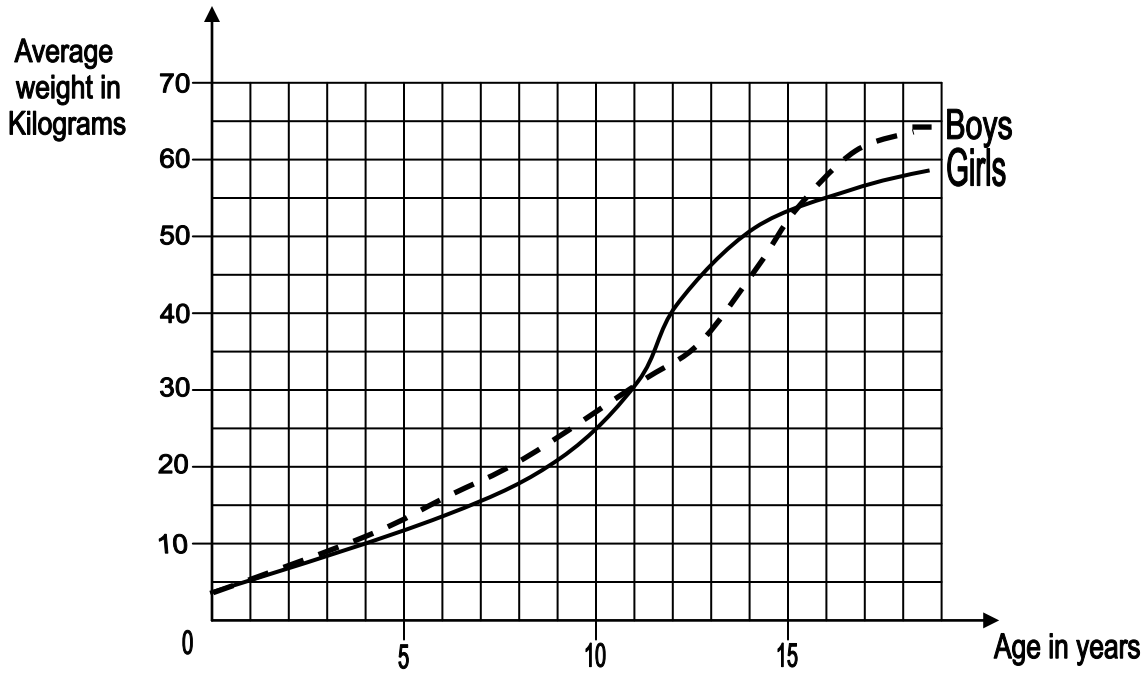


Figure 4.12: Diagnostic item on weight from the diagnostic test: average weight of boys and girls according to age.

All the teachers answered this question incorrectly. The following discussion took place during an interview:

Interviewer: This is the average weight of boys and girls in kilograms and the average age in years for boys and girls. You said the boys are growing faster?

Teacher: Because boys are taller than girls.

Interviewer: But does that mean that they are growing faster?

Teacher: Physical appearance shows they grow in height.

Interviewer: Does that necessarily mean that they have grown faster than girls just because they get taller than girls. They grow bigger but does that mean they grow faster?

Interviewer: Go to the graph, average weight versus. age in years.

Teacher: Girls weigh more than boys.

Interviewer: I weigh the same for 5 years, and now suddenly the boys weigh more than me? Who then is growing faster?

Teacher: It's the boys.

Interviewer: If the boys were weighing a certain weight and suddenly the girls shot past them who would be growing faster?

Teacher: The girls.

Interviewer: What do you understand by growing faster?

Teacher: It's a change.

Interviewer: In graphs how do we measure change?

Teacher: In graphs you measure change using vertical distance over horizontal distance.

Interviewer: Gradient?

Teacher: Yes.

Interviewer: Okay let's look at the graph again; what is the change indicated at around 14 years old for the boys?

Teacher: It's just a small change.

Interviewer: And the girls change? Which is smaller?

Teacher: This is the one (*indicates the boy's).*

Interviewer: So who is growing faster?

Teacher: It's the girls, sorry the boys.

Interviewer: Why?

Teacher: Boys undergo a faster change – its steeper.

A different teacher suggested that girls grow faster because "the line for girls is above the boy's line in the graph". This was the response from numerous learners. Another popular answer was 'but girls grow faster than boys', where clearly the context (prior knowledge/experience) interfered with their mathematical reasoning about the graphical representation.

The diagnostic test item in figure 4.13 also addressed a height-slope misconception while dealing with a graph representing speed. The respondents were asked to agree or disagree with Julie's observation that at 2pm the speed of car B is greater than the speed of car A, giving reasons.

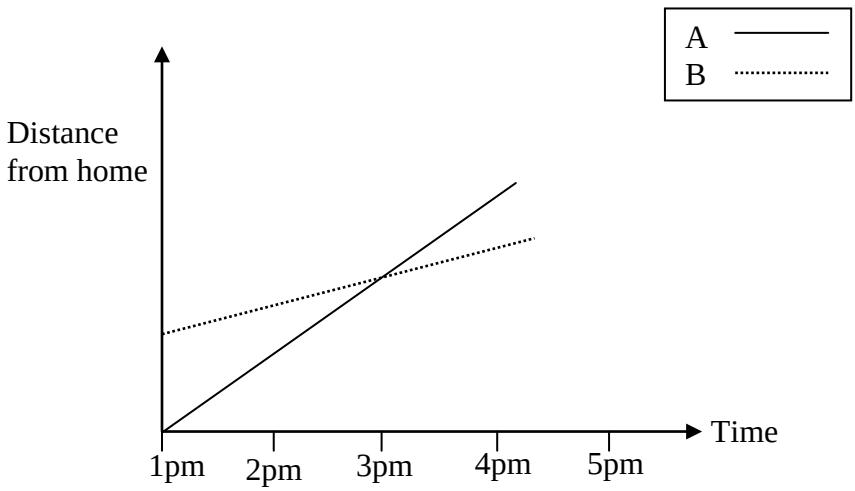


Figure 4.13: The distance of cars A and B from home.

Two other teachers agreed with Julie, stating that car B is higher than car A. The student responses indicated that height indicated speed and not the slope when they answered: “Yes, because speed of B greater than speed of A”; “Yes, because if you look at the line of car A it is past that of car B”; “Yes, because car B is higher than car A” and “Yes, because car B starts above car A so the speed of car B is greater than car A”.

4.2 Inability to interpret graphs (graphs as a picture)

In the diagnostic test the respondents were asked to look at the graph in figure 4.13 above on the distance of cars A and B and answer the question: What is happening to the two cars at 3 o’clock?

Teachers responses included that the cars were at the same place at 3 o’clock, at the same point (maybe at crossroads) or the routes intersect. Only one teacher gave the correct answer that the cars were the same distance from home. In the diagnostic test, most of the learners echoed the teachers responses that the cars met, crossed each other or had an accident.

The following statements were made by a teacher when discussing why the routes should intersect:

Interviewer: How do you know that the cars routes will be intersecting at 3pm?

Teacher: They are intersecting, crossing each other.

Interviewer: Okay, but does it mean that they are at the same place?

Teacher: No, it only shows that they have intersected at that point.

Interviewer: What do you mean by intersected?

Teacher: Their roads have come across each other.

Probing into another teacher's answer to the graph as a picture led to the preoccupation with the point of intersection of two straight line graphs. Repeatedly in the interview we returned to this aspect of the graph, which hindered the correct interpretation of the graph. The teacher responded as follows:

Interviewer: Meet? Why do you say they meet?

Teacher: This is a point of intersection,

Interviewer: Does it mean they are in the same place?

Teacher: They met at 3pm and are also at the same place because, look here they intersect.

The discussion continued and values were added to the axes of the graph; 10km for the y-intercept of car B and 15km for where the cars intersect. Speed was then discussed and we continued as follows:

Teacher: They meet each other at 3 o'clock; it is the point of intersection.

The teacher then started talking about displacement and distance which led to the following interlude:

Interviewer: But is distance travelled the same as displacement?

Teacher: No.

Interviewer: So does this graph show the cars meeting in the same place?

Teacher: But, there is a point of intersection.

A further discussion of how far each car has travelled over time and from where resulted in the following debate:

Interviewer: But if the cars were travelling in different direction i.e. North and South are they going to meet in the same place?

Teacher: No, cant.

Interviewer: So what does that point of intersection mean?

Teacher: (LONG PAUSE) The point of intersection, where the graphs meet.

Interviewer: Yes, so what does it indicate about that distance?

Teacher: But is not the same distance here (indicating the y axis). From zero and then it started at 10 km which means that this one was nearer to the position and this one was far. But at 3pm they intersect at the same place they are in the same place.

The teacher then realised that the graphs also indicate speed which is different for each car because of the gradients.

Interviewer: And what does that mean?

Teacher: Than means that car A is faster than car B

Interviewer: Okay but what does it mean in terms of the distance we have travelled?

Teacher: The same

Interviewer: It's the same distance, but if I went from Dimbaza at the same speed towards Alice and you went to King, at this point (3pm) would we meet?

Teacher: According to this graph at 3pm, the graphs intersect.

In terms of answering the diagnostic item with respect to Julie's question of which car was travelling faster at 2pm, learners indicated on their scripts that they interpreted the graph as a picture and gave the following responses:

"If the speed of Car B is greater than the speed of car A, why should the meet at the same road and at the same time at 3 o'clock? So that means that their speeds are equal."

"The Car B speed is smaller than Car A because the road of Car B is short"

Another diagnostic test item where the respondents mistook the graph for a picture was 'Peter's walk'. The question required the interpretation of the graph using one or two sentences to explain what was happening. An example had been done for the respondents before they were asked to interpret the question in figure 4.14, which shows the graph of Peter's walk.

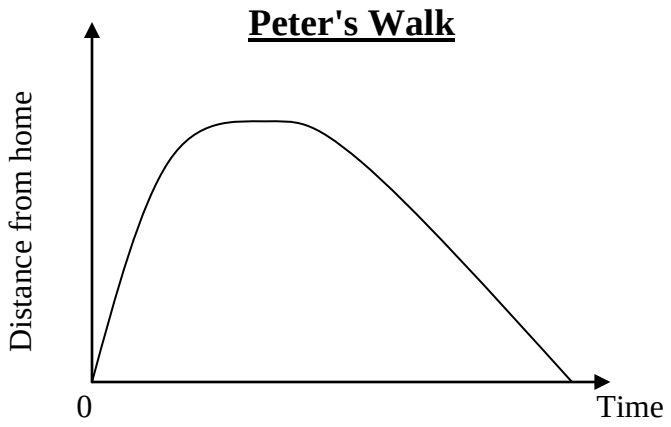


Figure 4.14: Peter's Walk

Some of the learners saw it as a picture of a hill, indicative in the following response: "Peter walk goes up, then the road is short and when it goes down it becomes long'.

4.3 Connections between graphs, equations and tables

During an interview, while discussing how different speeds are represented on a graph, a teacher fell back on the theory of x and y values, tables and equations whilst discussing the question: How would your represent speed on a graph? The teacher responds saying that it depends on the steepness of the graph, the faster the object is going the steeper (higher) the line towards the y-axis. Then the following dialogue occurred:

Interviewer: Ok then what does the x and y axis represent?

Teacher: XY axes, y axis on the graph represents how high?

Interviewer: How high what? How high can speed go?

Teacher: No, when we are looking at the speed, when you are doing a graph you must have a table with x and y. And then on y those are the variables and y depends on x you see, if x = 2 then y = 3 this shows that y depends x. [*the teacher draws a table and makes a sketch, figure 4.15 by way of explaining*]

x	2	
y	3	

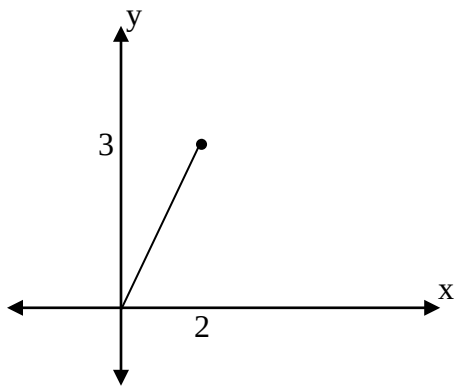


Figure 4.15: Teachers table and sketch for explanation.

Interviewer: But what does the y axis - I understand that you have variables x and y - but what are we plotting on x and y and how do we get those variables for speed?

Teacher: Ok, on the y axis we are plotting the outcomes if we are given the values of the x axis. Let's take for example if we are given y = 2x +1 then we are given the value of x and what will be the outcome of the equations.

The discussion then moved onto distance over time with the gradient representing average speed.

4.4 Gradient/rate of change

In part two of the diagnostic test the last non-contextual question looked at rate of change for the curves shown on the graph in figure 4.16. The respondents were first asked to explain which of the two curves had the greatest rate of change in the shaded interval. Secondly which curve had the greatest rate of change at the point where x = 25.

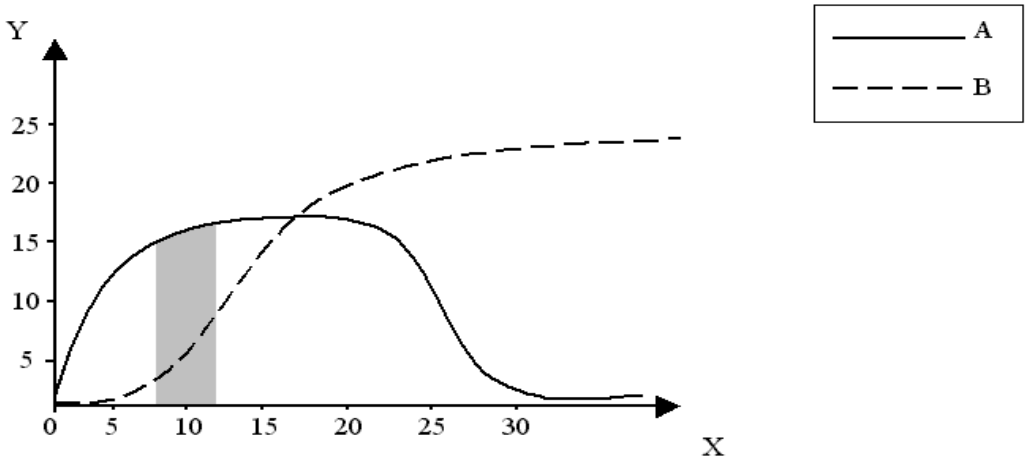


Figure 4.16: Diagnostic test item on rate of change.

Only one of the 66 learners answered this question correctly while the rest answered curve A for the first question because it was higher than curve B, once again confusing height and slope. Curve B was the solution given to the second question as curve B was now higher than curve A.

In considering what gradient was, the above question was scrutinized in a teacher interview. The following dialogue ensued clearly indicating that the formula for calculating gradient was known:

Interviewer: What do you understand by having the same slope or gradient?

Teacher: Change in y over change in x the same?

Interviewer: What is a constant rate?

Teacher: Gradient?

Interviewer: Does a gradient have to be constant?

Teacher: No.

Interviewer: Okay what is the constant rate of change?

Teacher: Change in y over x.

Interviewer: But I can get two different answers for the change of y over change of x in a graph.

Teacher: Same gradient.

Interviewer: What sort of graph do I have?

Teacher: Straight line.

Once it had been established that gradient was the change in the variables on the graph and constant rate of change was a straight line graph, we looked at the curves of the graph above.

Interviewer: What do curves represent on a graph?

Teacher: They mean that there is a change between x and y

Interviewer: At x = 25 which curve has the greatest rate of change? You said B, why?

Teacher: Look at the change in y over the change in x. [*goes on to try and calculate rate of change between 0 and 25*]

Interviewer: Okay but what about at 25? Not what happened before or after.

Teacher: Greatest change, at 25, it is [*still looking at the graph in confusion, looking at change over the whole graph?]*

Interviewer: Change at that instant. Which y is changing more?

Teacher: Graph A

Interviewer: How do I measure rate of change

Teacher: There is a formula, $\frac{y_2 - y_1}{x_2 - x_1}$, or you can use the derivative. [*the teacher then reverted back to calculus knowledge*]

Question 1 in the non-contextual section of the diagnostic test also addressed the issue of rate of change. It looked at the graphical representation of the relationship of the multiples of four as shown in figure 4.17.

Figure 4.17: Diagnostic Item: Multiplying by four.

The first question asked for the coordinates of point A. This question had the most correct answers by the learners compared to any other item on the test. Some incorrect answers were just cases of misreading of the y-coordinate by 1 value, possibly because of the given scale. However, when answering question two and three the number of correct solutions diminished. The second question asked that learners fill in the missing coordinates for (20,…) and (…,100). Learners mistakenly swopped the x and y coordinates around giving (20, 5) as the solution because 20 could be read off of the y-axis (it was not given on the x-axis). Following that they give a wide range of values for x, very few gave the right answer of 25. The values of x given for the second set of coordinates were lower than 100, which did not correlate with the constant gradient of four, with respect to their first coordinates or the relationship of multiples of four. Lastly respondents were asked to calculate the gradient of a straight line graph that represented the relationship of multiples of four. The learners demonstrated that they knew how to calculate the gradient using the correct formula but found the incorrect answer as they could not read off

points from the given graph. Strangely, one of the learners listed the formula for a straight line $y = mx + c$, as the gradient of a straight line.

In the contextual part of the diagnostic test the question relating to Mary and John's journey to the cinema investigated the graphical understanding of slope. The question asked to draw John's journey on the same graph on which Mary's journey was drawn (figure 4.18). The information where Mary first ran for 15 minutes and then walked for 20 minutes, was reversed, John first waked and then ran but at the same speed as Mary.

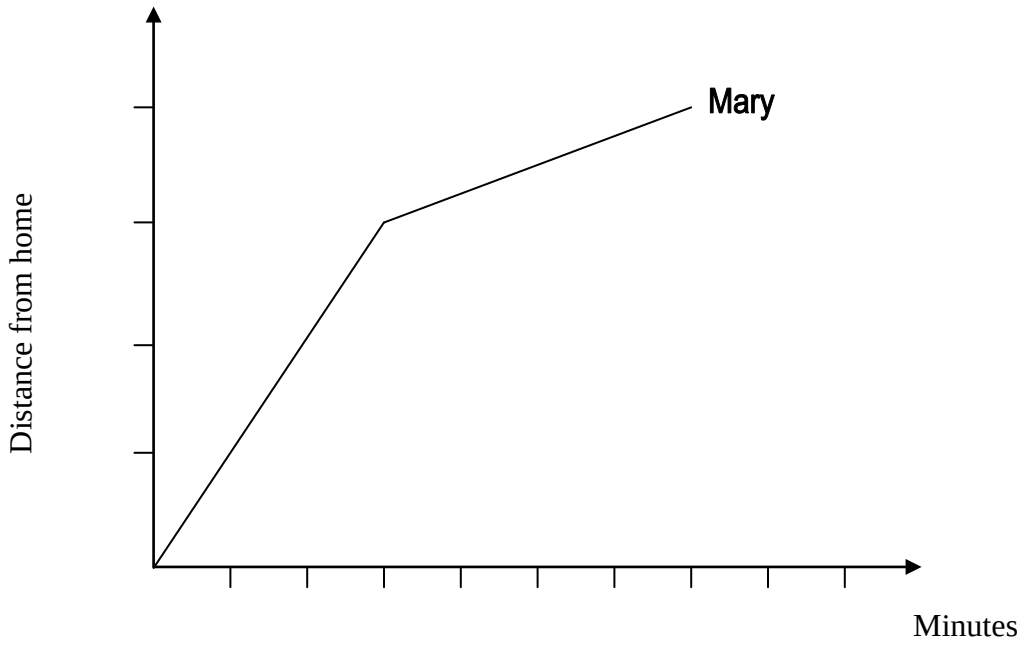


Figure 4.18: Mary and John's journey to the cinema

One of the teachers drew John's journey as shown in figure 4.19. The time intervals were correct but the gradients did not agree with the information given.

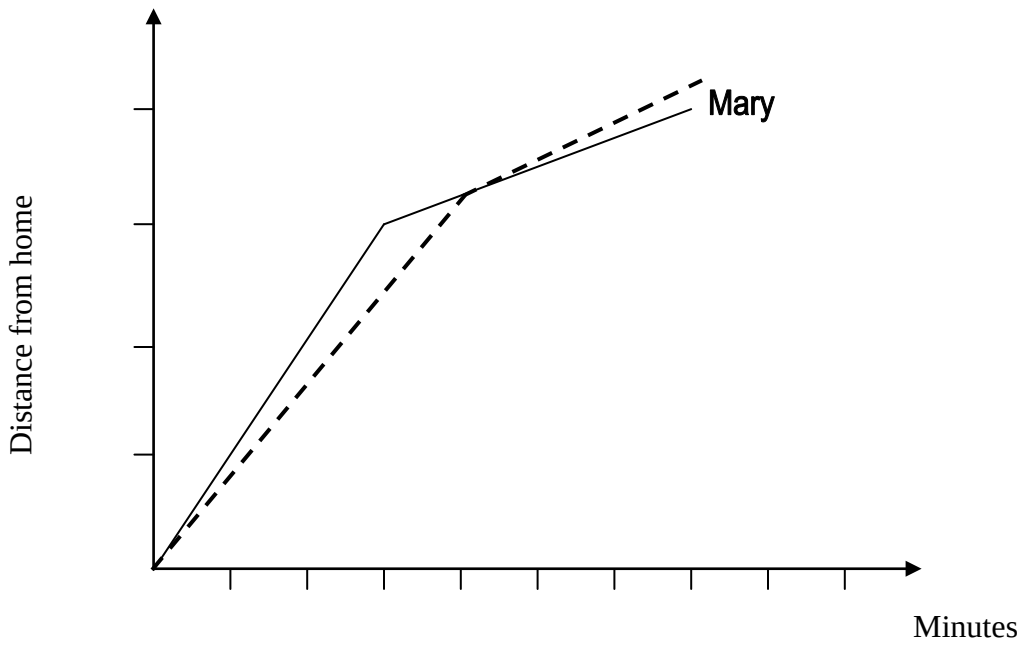


Figure 4.19: A teacher's solution to Mary and John's Journey.

During an interview with the teacher the following interactions took place:

Interviewer: What do you understand by the same speed? How do we represent same speed?

Teacher: Cover the same distance over a certain period of time.

Interviewer: If I look at the graph it represents distance over time, so how would I represent the same speed?

Teacher: How long the distance is in relation with the time?

Interviewer: If John runs for the last 15 minutes when does he start running? At what time.

Teacher: At 20 minutes.

Interviewer: If he is running from the 20th minute, how would you draw that on the graph, remembering that he runs at the same speed as Mary?

Teacher: I would draw it as I showed here but past the 15 minutes Mary was…. *[Indicates on the graph at 20 minutes]*

Interviewer: Okay but what did John do first?

Teacher: John walks for the first 20 minutes.

Interviewer: Okay when does Mary walk?

Teacher: She runs for the first 15 minutes and then runs for the next 20 minutes.

Interviewer: Okay now John walks for the first 20 minutes, why have you drawn that he walks as fast as she runs? What do you understand by slope?

Teacher: Steepness? The slope and steepness, shows us the gradient of the graph?

Interviewer: Okay so why have you drawn John's gradient steeper than Mary's? Why are the gradients different if they are going at the same speed?

Teacher: The one for the 15 min is steeper than for 20min because maybe the gradients are not the same.

Interviewer: But why would the gradients not be the same if they were travelling the same speed? Which one of these is going faster?

Teacher: This one. *[Indicates the less steep gradient]*

4.5 Language and misconceptions

In the contextual item on transport, figure 4.20 and figure 4.21, the y-axes were labelled distance travelled in km and distance to school in km respectively. Respondents were asked to draw the second graph which looked at distance to school from home in km according to the first graph where the distance travelled to school from home in km was given by line C.

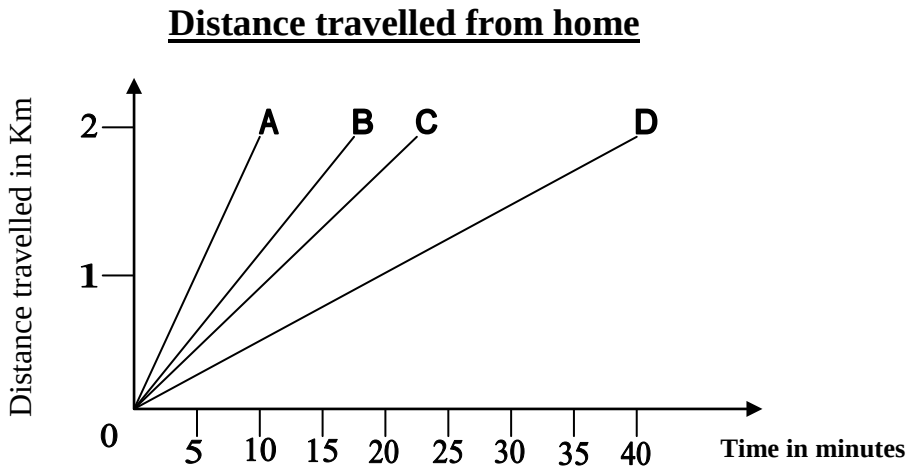


Figure 4.20: Diagnostic test item on transport: Graph 1

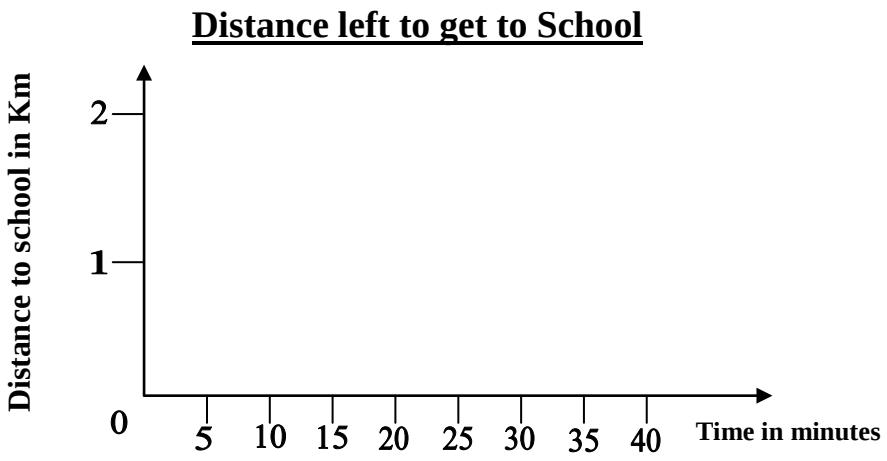


Figure 4.21 Diagnostic test item on transport: Graph 2

One of the teachers drew exactly the same graph C, as given in Graph 1 (see figure 4.20), failing to notice the nuance of the label on the y-axis that changed the question. In exploring the question together the following ideas emerge:

Interviewer: What is the difference between distance travelled and distance to?

Teacher: Distance to school is what is left to school, the distance that a person has to go to get to school. In terms of distance, it is getting shorter. I mean that you are getting nearer and nearer to the school.

Interviewer: That's right but now your graph is telling me I am getting further and further.

Teacher: Here now, it is in terms of km being travelled, that you have past 1km, 2km.

Interviewer: Okay but is that not distance travelled?

Teacher: Yes.

Interviewer: You told me that ‘distance to’ is what is left to school.

Teacher: [*uncertain, hesitates*] It's the language.

Interviewer: Let's talk about this language. You said ‘distance travelled’ is what I have covered and ‘distance to’ is what I have left to get to school. So the first graph shows distance travelled from home to school and the distance increases as time passes. The second graph which you are asked to draw is distance travelled to school in km, so as I go to school my distance has to…..?

Teacher: Decrease.

Interviewer: Okay, so as time moves on I get closer to school. So how would we draw this graph? So at 0min I am 2km away. So as I get closer the distance becomes?

Teacher: Less and the graph needs to decrease. [*draws the graph correctly making it decreasing*]

4.6 Scale

Learners and teachers alike did not enjoy dealing with labelled axes where no intervals or values were given. When answering these questions they inserted their own intervals and numbers to assist in making sense of the questions. In the diagnostic item on charity in part one

of the test two learners created intervals for the labelled axes but used an inconsistent scale where the values did not match the space between their marked gaps. One learner marked the axes as shown in figure 4.22 showing that the ticket prices for two people was R10, making the cost per person R5, and then five people paid R15 where the cost became R3 per person and continually the price per person changed as you read off different points of the graph. This change in price per ticket contradicted the drawn graph which had a constant gradient which meant the same rate of change. Hence the values of the intervals given by the student were inconsistent with a constant ticket price.

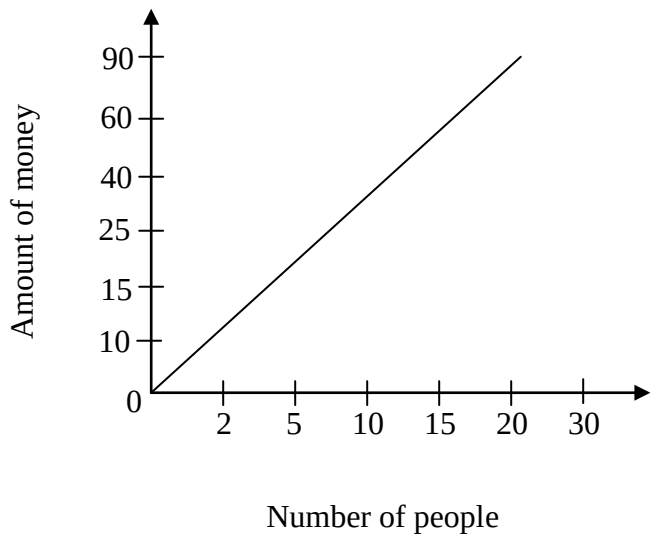


Figure 4.22: Charity graph where learner labelled the axes

4.7 Origin prototype

Four out of the five teachers held the origin prototype misconception, the origin is seen as a prerequisite of a graph by starting to draw the graph through it. This was evident when they were asked to graph the height of a person from birth to age thirty. The teachers drew the height of the person at birth to be 0m. An example of one of the teacher's graphs is shown in figure 4.23.

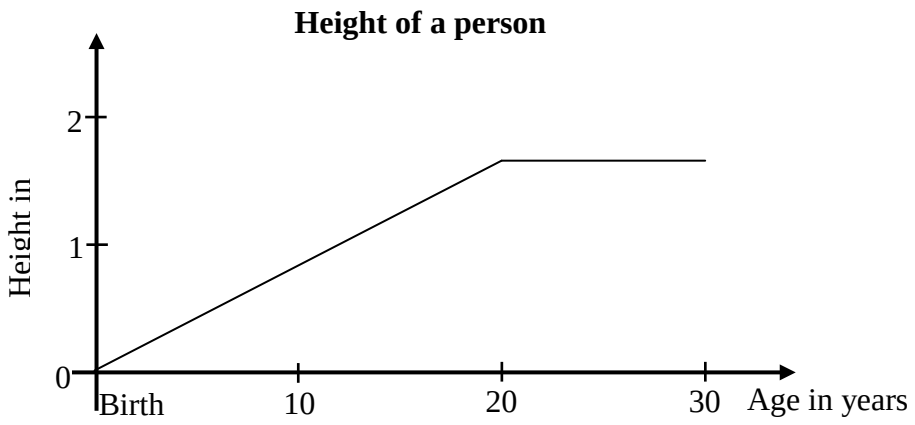


Figure 4.23: Teacher's graph of the height of a person

In an interview with a teacher the notion of the height of the person being 0m at birth was challenged:

Interviewer: Okay, let's look at the start of your graph. From your graph tell me what is a person's height at birth?

Teacher: 0m, no, that can't be, that means you don't exist. *[confused]*

The teacher realised that the graph should not go through the origin but have and intercept of 50 – 60cm.

The few learners who drew straight lines also fell into the trap of drawing their graphs through the origin. However, a majority of the learners who attempted this question mistakenly drew pictographs or bar graphs indicating that 10 year olds were shorter in height than 20 and 30 year olds respectively.

In the question discussed in 4.2 about cars A and B a surprising answer from one of the teachers was "The speed of car A is faster because it is travelling in a straight line, and that of car

B is not straight". Car A started the trip at the origin while car B did not go through the origin but had a positive y-intercept.

5. CONTEXTUAL QUESTIONS AS A MEANS OF REMEDYING MISCONCEPTIONS

The weight question as discussed in 4.1 above was answered very poorly by the learners. Most of them ignored the information given in the graph and relied solely on the context of their experience to answer the questions. The two questions asked with reference to the graph were:

1. What is the difference between boys and girls weight at age 14?
2. Which group is growing faster at the age of 14? Explain your answer.

When answering question 1 most learners failed to give an amount that indicated the difference in the boys and girls weight. Their answers correctly took into account that the line of girls' weight was above that of the boys, but they failed to quantify the difference. Question two was answered in such a way that the graph was ignored and their contextual experiences lead them to answer incorrectly, i.e. that girls grow faster. A few of their explanations were that girls gain more weight than boys at the age of 14; at the age of 14 girls weighed 50kg and boys only 45kg, therefore the girls are growing faster because they are heavier. Two of the teachers' indicated in their questionnaires that they held different opinions on the use of context; one felt that the context would hinder while the other felt the context could assist the learners' ability to answer this question.

The following is an excerpt from a discussion with a teacher on the same question:

Interviewer: What does weight have to do with growth of a person? If a girl is heavier does it means she grows faster?

Teacher: In real life girls are growing faster, until the stage of 16 -20 years.

Interviewer: Okay so this is for the age 14, where is the age 14? Where would I read off the figures?

Teacher: [*The teacher indicates the x-axis*] Boys 45kg and girls 50kg

Interviewer: Ok, what were the weights at 13?

Teacher: Boys 37kg girls 46kg

Interviewer: Ok, so by how much has each changed?

Teacher: Boys 8kg and girls 4kg.

Interviewer: So who gained more weight from 13 to 14?

Teacher: Boys.

Interviewer: But you said the girls are growing faster?

Teacher: At what?

Interviewer: At age 14. Just because the girls are heavier does that mean that they are growing faster?

Teacher: No.

Interviewer: So who is growing faster?

Teacher: But girls grow faster than boys.

Interviewer: That may be what you think in reality, but reading the information off these graphs, these are now the facts we are looking at.

Teacher: The boys.

Interviewer: Why?

Teacher: The boys look at the average.

Interviewer: The average of what?

Teacher: Change in y over the change in x

Interviewer: Which is the?

Teacher: Gradient

6. SUMMARY

This chapter recorded the results of the study. This was done by documenting the learner and teacher responses to the diagnostic tests by taking into consideration the percentage of correct responses in each group according to the two groups of contextual and non-contextual sections of the test. Then the teacher-learner scores were compared by firstly contrasting each teacher's mean score with their learners' group average. The scores were then compared collectively in terms of the teachers' results against the learners' performance.

The teacher's qualifications along with their years of teaching experience were recorded. Each teacher's predication of the difficulties their learners would experience was noted and then compared to the teachers' estimate of the number of learners who would correctly answer each question with actual learner performance. The identification of the errors and misconceptions learners would experience when completing the diagnostic test was tabulated along with how the teachers proposed to intervene and address these problems and misconceptions.

Excerpts from the transcribed teacher interviews, along with teacher and learner responses in the diagnostic tests and teacher questionnaires, were examined and common misconceptions in graphicacy were identified. Issues around the use of contextual and non-contextual questions in order to aid understanding were also examined.

CHAPTER FIVE

DISCUSSION

1. INTRODUCTION

In this chapter the results which are presented in chapter four are analysed in terms of the research questions posed in chapter one. The question around the teachers' content knowledge is examined with respect to the diagnostic test, questionnaire and in-depth interviews they completed. The second question addressed is the question of the level of content knowledge held by the learners' as reflected by the results of their diagnostic tests. Common misconceptions held by the teachers and learners alike, as well irregularities found in all the data sources are addressed in terms of possible reasons for their existence against what previous studies have reported. The level of pedagogical content knowledge held by the teachers is considered via the comments they made when completing the questionnaire and during the interview process. Finally, all of the above are brought together by looking at what the nature of the relationship between the teachers SCK, PCK and commonly held misconceptions, and how these aspects could affect learner achievement. In attempting to answer each question appropriate data is considered from all three sources where applicable, viz. the diagnostic test, questionnaire and interview data.

2. WHAT IS THE LEVEL OF CONTENT KNOWLEDGE HELD BY THE TEACHERS?

As noted earlier, the diagnostic test, questionnaire and interview data are considered where applicable.

2.1 Diagnostic tests

The teachers scored an average of 58% on the test overall, a level of mastery that probably most mathematics educators would deem as being an insufficient understanding of graphicacy if one is to teach the topic successfully. Although the teachers fared better in the non-contextual questions in the test compared to the contextual section, there were obvious problems with their content knowledge in both sections. There were three questions that none of the teachers got right, two in the contextual section and one in the non-contextual section of the test. These issues are discussed in greater detail below.

2.1.1 Contextual questions

The fact that the teachers fared poorly in section (b) of the transport question may possibly be attributed to issues of language as the question used the words ‘distance travelled’ in part (a) and ‘distance to school’ in part (b) as Hart (1981) sees the language used in mathematics lessons as being technical, which means it differs from learners’ regular vocabulary and therefore often need to be redefined. Another possible explanation, based on mathematical understandings, could be that they were required to draw a negative gradient but continued to visualise the graph in terms of the positive linear model of rate of change, as noted by Heid, et al. (2004).

In the second sub-question in the weight question, the teachers' poor performance could be attributed to the slope-height misconception and the influence of a different perceived reality of the context. For instance, because the teachers initially perceived that girls weighed more than boys this caused them to concentrate on the context at the expense of the question, despite the greater rate of change shown in the boys' graph. Blinko (2004) suggests that contextual questions make abstract ideas more accessible to learners. However, this contextual question appears to have misled the teachers as their understanding of the given context was different to the facts which were presented; hence the context did not assist in countering slope-height misconception. Julie, et al. (1998) and Kerr and Maki (1979) see contextual questions as appropriate only if the simplification of the context and mathematical application thereof is accurate, otherwise it detracts from the mathematical ideas present.

2.1.2 Non – contextual questions

Question 6 dealt with the greatest rate of change of two curves, the first part in respect to an interval and the other to that of a fixed point. Teachers once again looked at the height of the graph and not the gradient at the point given. Only one teacher got the rate of change for the interval correct and none answered the rate of change at the given point correctly. This reinforces what Crawford and Williams (2000) found in their research on gradients, i.e. that while learners seldom have difficulty calculating the gradient; they struggle to understand its meaning. Calculating a gradient over an interval when given points and the formula is easy. However, the problem remains in understanding how variables change with respect to each other as they get closer to a point.

2.2 Teacher questionnaires

The demographics of the teachers' qualifications and years of teaching experience are recorded in table 4.1 in chapter four. The highest qualification held by the teachers was diploma in education. The tertiary education that teachers underwent would have included some form of mathematical SCK, along with some mathematical knowledge that they had picked up from school. Ball and McDiarmid (1990) suggest the interaction that teachers have with mathematical SCK is often limited to what they come across at school, and then to a limited introduction in their professional training at a tertiary level. Nevertheless the amount of SCK that is covered in training is questionable (Ball & McDiarmid, 1990), which appears to be the case in this study.

At the time of data collecting during the study the teachers had completed a year and a half of a three year teacher upgrading project. As part of the project they were improving their qualifications to a BEd FET degree in mathematics education, which required them to complete first and second year pure mathematics courses and a number of method courses. Therefore, over and above their schooling and initial tertiary teacher training, they had most recent exposure to attempts to extend their mathematical SCK, yet they still performed poorly in the diagnostic test showing little evidence of substantial SCK development.

2.3 Teacher interviews

The teacher in-depth interviews considered some of the ambiguities picked up in the teacher diagnostic test and teacher questionnaires in the context of the graphical misconceptions identified by previous research (Coulombe & Berenson, 2001; French, 2002; Goerdt, 2004; Kerslake, 1981; Van Dyke & White, 2004). These misconceptions include slope-height confusion (Leinhardt, Zaslavsky & Stein, Hadjidemetriou & Williams, 2002; 1990; Miranda,

2007); inability to interpret graphs, particularly as seeing the graph as a picture (Bell & Janvier, 1981; Clement, 1985; French, 2002; Heid, Zbiek & Blume, 2004; Janvier, 1987; Kerslake, 1977, 1981; Miranda, 2007); problems relating to rate of change (Coulombe & Berenson, 2001; Crawford & Williams, 2000; French 2002; Heid, Zbiek & Blume, 2004); misconceptions associated with language (Hart, 1981; Dias, 2000;); origin prototype (Ryan & Williams, 2007); and contextual questions as a means of remedying misconceptions (Blinko, 2004; Kerr & Maki, 1979; Williams, 2007).

Swan (2001) notes that some of the responses which are viewed as misconceptions may appear to be misconceptions due to the respondents being too hasty and losing concentration. However, Goerdt (2004) places them in a mathematical context, e.g. teachers not being able to identify which important aspects of the graph to focus on, such as gradients and intercepts. The interview dialogue and written results and comments, as recorded in chapter four, suggest that the graphical misconceptions being investigated were due to not being able to identify and focus on important aspects of the graph, as postulated by Goerdt (2004).

3. WHAT IS THE LEVEL OF CONTENT KNOWLEDGE HELD BY THE LEARNERS?

Learner and teacher achievement in the diagnostic test, teacher questionnaire and interviews, are considered below in an attempt to answer the question above.

3.1 Learners' diagnostic tests

The learners' diagnostic test showed an overall low competence in dealing with graphs (see figures 4.1 and 4.2 in chapter four). Their collective average of approximately 20% could be

attributed to little previous knowledge being built or developed because of the teachers' poor educational background, largely as a result of the educational system that operated when the teachers were trained (Hartshorne, 1992). Morrow (2007) suggests there are many schoolteachers from that system who do not display the competencies to embrace and implement the required paradigm shift to implement the new curriculum.

One of the aims of the project was to engage with teachers to improve the mathematical competency levels through the upgrading of qualifications and instructional workshops. Despite these interventions the teachers performed poorly with their learners producing even weaker results. The poor abilities of the teachers would work against the development of what Kilpatrick, Swafford and Findell (2001) describe as 'an organised knowledge system in which new theory connects to pre-existing knowledge' in their learners. These teachers would most probably find difficulty in promoting the interpretation processes that Janvier (1987) argues are generally neglected in teaching approaches to graphs, and would be most likely to focus on the three representations of table, equation and graphical representation.

3.1.1 Contextual questions

An anomaly was found in terms of some of the answers to the contextual questions in the diagnostic test. The transport question was well answered by learners with approximately 80% giving the correct answer compared to 50% or less in the remaining items. The item required learners to associate the travel times of different modes of transport with drawn graphs of distance over time. However, the high level of proficiency in this question could be attributed to the fact that the first part of the question had been done for them by means of an example illustrating how they were required to answer the question. Also, in this item, they did not have

to work with what Hadjidemetriou and Williams (2001) call ‘sophisticated graphical understanding of slope’, but could simply read off the time taken in each graph and associate it with the different modes of transport. Context could also have played a role as learners could relate to the fact that in South Africa a bus would probably travel more slowly than a car would, and that walking would take the longest, contextual clues which Blinko (2004) suggests direct the learner into what the appropriate solution should be.

Three questions which failed to elicit any correct solutions by the learners were cars (a), story (a) and charity (b). Cars (a) and story (a) both probed the ‘graph as a picture’ misconception, and learners answered both items as if the graphs represented pictures. Clement (1985) and Kerslake (1977) describe this misconception as being due to graphs seen as literal pictures, i.e. as something which hinders the correct interpretation of the graph.

The question cars (a) asked learners to interpret the point of intersection of two journeys from home. The majority of learners responded that they were at the same place or having an accident, ignoring what happened after the point of intersection instead of interpreting them as being the same distance from home. The learners failed to see that their answers were not sensible when looking at the nature of the graphs after the point of intersection and were unable to workout the correct interpretation. Again, this error could be ascribed to the fact that learners saw the two graphs as pictures (Kerslake, 1977; Clement, 1985) or that they, like one of the teachers, only focused on one of the important features of the graph, i.e. the point of intersection (Goerdt, 2004) and neglected to note the relationship between the variables ‘distance travelled from home’ over ‘time’ which actually represented the speed of the cars.

Story (a) required learners to describe Peter's Walk in one or two sentences with respect to the distance from home over time. The learners spoke about the hill he climbed and what time it took but neglected to mention anything about the distance from home indicating that the graph was seen as a picture. Kerslake (1977, 1981) found that learners often interpreted distance-time graphs as 'going uphill then downhill, then up again' because a picture is seen rather than a relationship between the variables of distance and time.

3.1.2 Non – contextual questions

Learners attained very poor results for the non-contextual questions of the test, with no question receiving more than 50% correct responses. Many of the non-contextual questions focused on the concept of gradient (rate of change). Crawford and Williams (2000) in their study found that calculating slope using a given formula is easy enough for learners but that they seldom understand what it represents. When answering question 1(c) a learner quoted the straight line formula instead of calculating and explaining how to find the gradient. It is evident that the learner did not understand what the gradient was or what the equation $y = mx + c$ symbolised, but was memorising formulae and repeating them by rote. In memorising the standard equation of a straight line the learner does not achieve what Sierpinska's (1994) classifies as the first level of the four mental operations to understanding. Level one refers to identification whereby the learner is capable of focusing attention to a concept while naming and describing it. Carpenter and Lehrer (1999) talk about compartmentalised knowledge, the learner knew the formula was somehow related to gradient but unable to demonstrate the meaning of the equation or the meaning of the gradient.

Question 2(c) was one of the best answered questions by the learners in the non-contextual part of the test. It required them to determine if the graph showing the relationship of squaring numbers had the same slope everywhere. The learners who gave the correct answer said no because it is not a straight line. None of the learners showed Kilpatrick, Swafford and Findell's (2001) in-depth understanding of knowing more than the isolated fact that a straight line has a constant gradient rather than the slope in question 2(c) was not the same because the variables were changing at a different rate to each other (Heid, Zbiek & Blume, 2004).

In questions 4, 5 and 6, ten percent or fewer learners attained the correct solutions. These three questions focused mainly on the understanding of gradient. Learners were not required to calculate the gradient but to recognise constant gradient; rank slopes from least to greatest and answer questions on rate of change related to curves. In question 4, six different types of graphs were given and respondents were asked to identify which of the graphs showed that y was changing at a constant rate. For question 5 four line graphs were drawn on the same set of axes at different slopes and learners were first asked to arrange the graphs in order of least slope to the greatest and then explain their order. Question 6 dealt with gradient of curve graphs both over an interval and at a point. Learners were asked which of two graphs had the greatest change in both cases of the interval and the point. Heid, Zbiek and Blume (2004) suggest learners struggle with the concept of gradient because in classrooms teachers do not differentiate between rate of change and amount of change.

4. WHAT COMMON MISCONCEPTIONS IN TERMS OF GRAPHS ARE HELD BY THE TEACHERS AND THEIR LEARNERS?

Cramer (2004) found that teachers are often embarrassed by lack of mathematical SCK, while other research (Even, 1990; Hershkowitz & Vinner, 1984; Linchevsky & Vinner, 1989; Mostorides & Zachariades, 2004; Ponte, 1985; Sánchez & Llinares, 1991; Tall, 1996; Taylor & Vinjevold, 1999; Tirosh, Graeber & Glover, 1986) suggest that many teachers have gaps and distortions in the understanding of mathematical concepts. Linchevsky and Vinner's (1989) study found that the likely misconceptions anticipated from work with children were exactly the same as those held by their teachers'. Analysis of the diagnostic test, questionnaire and interview data confirmed that in this study the teachers' and learners' subject content knowledge have definite weaknesses, that many common misconceptions pertaining to graphicacy were held by both the teachers' and their learners', and that the learners' understandings mirror those of their teachers.

We shall look at some of these misconceptions in more detail below. There are items where more than one misconception is illustrated.

4.1 Slope-height confusion

Teachers and learners could not distinguish between the implications of the height and gradient of the graph. Research has shown that when considering rate of change the focus is on the height of the graphs rather than the relational change between the variables, i.e. the gradient (Bell & Janvier, 1981; Clement, 1985). In the contextual question about the average weight of boys and girls as they grow, the girls were heavier than the boys at age 14, but the learners

answered that the girls grew faster than the boys because the steeper gradient of the boys at the age of 14, indicating the change in weight, was ignored.

In an interview with a teacher it was only when the graph was discussed with respect to the change in weight in relationship to age that gradient was considered instead of which graph was the highest at age 14. This is what Hadjidemetriou and Williams (2000) described as a pointwise versus a variational approach when the value at a point distracts from the increase. Table 5.1 looks at these approaches for the diagnostic item on weight. The pointwise approach takes into consideration the coordinates of the weight of boys and girls respectively at corresponding ages i.e. the height of the graph at the specific ages. While the variational approach looks at how weight varies over time and how steep or great the change of the graph is between the ages, i.e. the gradient.

Pointwise

Age	Boys	Girls
13	37kg	45kg
14	45kg	50kg
15	53kg	53kg

Variational

Figure 5.1: Pointwise and variational approach for diagnostic item of weight

When looking at the cars problem the slope-height misconception is once again evident as most of the learners struggled to answer which car had the greatest speed. Because the focus was pointwise instead of variational (Hadjidemetriou & Williams, 2000), most of the learners

looked at which car was furthest from home at 2pm, i.e. the highest value, rather than the variation in the relationship (Heid, Zbiek & Blume, 2004; Leinhardt, Zaslavsky & Stein, 1990; Miranda, 2007) between distance and time, which is the speed represented by the slope of the graph.

4.2 Inability to interpret graphs (graphs as a picture)

In the cars question, the appearance of the graph distracted from the interpretation of the graph because of its spatial representation (Kerslake, 1981). The teachers and learners both saw the cars as crossing or meeting each other at the 3 o'clock point. This indicates that they are unable to treat the graph as an abstract algebraic relationship and interpreted the point as a literal picture (Miranda, 2007), where the graph crossed the roads must be intersecting. Specifically they misinterpreted what the point of intersection of the graph represents as well as the features of the graph which precede and follow the point of intersection, which Clement (1985) describes as a feature correspondence error.

The learner's responses that 'speed of car A and B were the same otherwise they would not meet at 3pm' and 'the road is shorter for car B therefore the speed must be slower' reinforced the fact that the graphs were seen as a map of a journey (French, 2002). The learners were looking at the given graph as the picture of roads crossing, and not the relationship between, distances from home over time. Similar results were found in the studies of Clement (1985), French (2002) and Kerslake (1977), where learners saw graphs incorrectly as literal pictures which hindered the correct interpretation and meaning of the graph because they failed to comprehend the abstract relationship between the variables (Miranda, 2007).

One of the teacher group discussions demonstrated an obsession with the point of intersection, where they returned to this aspect of the graph several times, continually misinterpreting its meaning. The teachers could identify the graphical feature which Goerdt (2004) identifies as important, and should be the focus because the intercepts are salient points, however they failed to interpret its meaning correctly (Van Dyke & White, 2001).

Both of the above examples compared two sets of graphical relationships on the same set of axes, which may have helped lead to the interpretation of graphs as a picture more readily. However the diagnostic test item on Peter's Walk shows one graph on a set of axes, yet still the graph is perceived as what French (2002) refers to as a map The statement "Peter walk goes up, then the road is short and when it goes down it becomes long", reveals how literally the graph is interpreted as a picture rather than a representation of the relationship between distance from home over time. As does the response in the teacher questionnaire that slope is "just like where you walk up a hill". These interpretations by teachers are most likely to exacerbate learner's natural inclinations to see graphs as a picture, as no algebraic relationship are emphasised by the teachers when teaching their learners about graphs (Miranda, 2007).

4.3 Connections between graphs, equations and tables

One of the teachers brought into the discussion the topic of connectivity between tables, graphs and equations when discussing how different speeds are represented on a graph. The teacher constructed a set of XY axes, a table for input and output values and an algebraic equation. This teacher clearly new there was an equivalence relationship (Van Dyke & Craine, 1997) between these three representations but, as noted by Janvier (1987), still ignored the critical step of interpreting graphs during her teaching.

4.4 Gradient/rate of change

When discussing the rate of change, learners and teachers alike were very comfortable in citing that gradient addressed the change in y over the change in x or gave the formula used to calculate gradient. Crawford and Williams (2000) found that learners were happy to define slope as the relationship between the vertical and the horizontal change, i.e. rise over run. However, very few see it in terms of how the variables of a graph change with respect to each other, something which French (2002) describes as a crucial feature.

When confronted with finding the change at a given point compared with the change over an interval some respondents again fell into two traps. Firstly that of confusing the highest value as the greatest rate of change because they were using what Hadjidemetriou and Williams (2000) suggest is a pointwise approach instead of the variational approach, as explained above. The second trap looked at the failure to understand that there can be a slope at a point and the formula needs two points with which to calculate rate of change (Hadjidemetriou & Williams, 2002). During an interview one of the teachers tried to calculate the rate of change at a point, i.e. $x = 25$, by calculating the gradient using the formula over the interval 0 to 25.

Crawford and Williams (2000) found it common that learners can calculate slope easily enough given the equation but seldom understand what it represents. The graph representing the relationship of the multiples of four showed that learners knew how to calculate gradient using the formula as long as they had two sets of points with which to work. In answering the question learners read off the coordinates incorrectly but used the formula accurately. Learners did not display any understanding of what gradient meant as they did not realise that the gradient they calculated was incorrect because they ignored the relationship between the multiples of four

where the constant gradient should have been 4. The learners could not identify any errors, which supports Crawford and Williams' (2000) notion that learners rarely know what gradient represents.

Learners also switched the x and y values of the coordinates. Possibly because in question on multiples the value for x = 20 was not available on the x-axis of the graph, but the value of 20 was situated and easily read off the y-axis of the graph. The learners ignored the convention of writing coordinates in the order (x;y). The reversing of axes order is something that has been also reported by Hadjidemetriou and Williams (2001).

The question on Mary and John on a journey to the cinema dealt with the graphical representation of the same speed at which the gradients had to be drawn parallel to each other to indicate the same speed for running and walking, but at different times. Crawford and Williams (2000) found that learners often failed to see that the same speed meant parallel lines as the ratio of the relationship between vertical change and horizontal change had to be the same. Learners and teachers struggled to understand this graphical representation of slope. In an interview with a teacher the speed relationship was seen as distance over time but not as the graphical representation of slope.

4.5 Language and misconceptions

Hart (1981) suggests that mathematical language is often technical and needs to be redefined in the classroom. Dias (2000) also contends that common usage and meaning of words are different when used in mathematical situations and should be clearly redefined. However, this aspect of language use and understanding could have been exacerbated as the teachers and learners in the study were not mother-tongue English speakers, they may have struggled to

understand the questions and/or instructions because they could not grasp or comprehend all of the English words and terms.

However, only three or four of the recorded responses from the teacher questionnaires in indentifying errors and misconception or in rectifying approaches highlighted the importance of language and the understanding of terminology used. In an interview discussing the transport item (b), where one was asked to draw the graph of the distance left to school over time according to the information given in graph (a) if you were walking, a teacher mentioned that 'it's the language'. This could be due to the idiosyncratic language used of 'distance travelled' in the first graph and 'distance to' on the y-axis in (b).

4.6 Scale

In the question on charity learners were told that each person who attends a charity concert pays an entrance fee, so the more people who attend the more money will be collected. The learners were then asked to explain why the given graph goes through the origin. One learner appeared to ignore the question and inserted his own scale where axes were named and the scales given were at irregular intervals, i.e. scaled unevenly. Even though their intervals distances were standardised (conventional), the numbers given were inconsistent and varied considerably, a strategy noted by Surtrees (2005). Another learner also produced an inconsistent scale and neglected to note what effect this had on the gradient. A straight line was drawn but the scale values that the learner filled in against the axes were non-linear and contradicted the straight line graph that was presented, i.e. the axes figures did not produce a constant gradient. These learners failed to see what French (2002) describes as the crucial relationship of how the variables change with respect to each other and ignored the fact that the amount of money was

dependent on the number of people. When one looks at a table created from a learner's values and what they represent, as shown below in table 5.1, one clearly sees that constant rate of change is ignored because the cost per a ticket is inconsistent.

Table 5.1: Table showing the inconsistent intervals given by a leaner in the charity item.

People (x)	**2**	**5**	**10**	**15**	**20**	**30**
Amount of money (y)	10	15	25	40	60	90
Cost per ticket (R)	5	3	2,50	2,67	3	3

If one was to sketch the graph of the learner's values against a correctly scaled graph the result would be the graph in figure 5.2, which does not show the constant rate of change as was given.

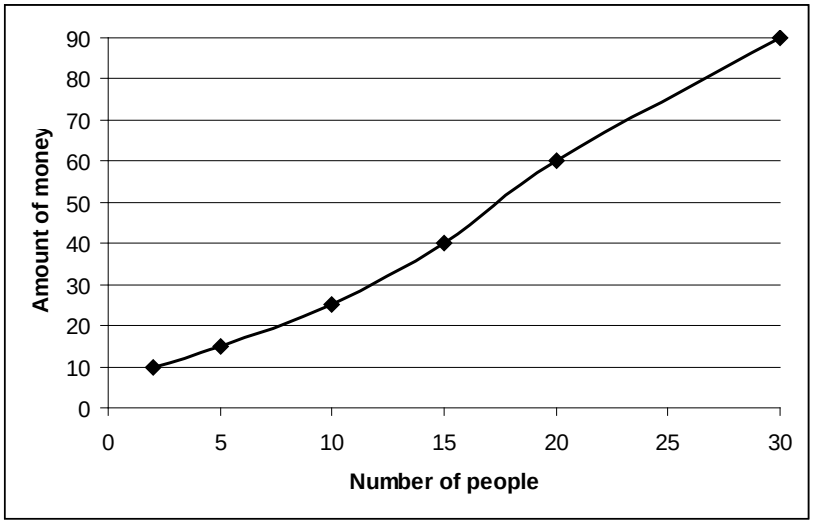


Figure 5.2: Graph of learners values for charity question

4.7 Origin prototype

The origin prototype is seen as the need to draw a graph through the origin (Hadjidemetriou & Williams, 2002). When teachers were asked to draw the height of a person in

metres over age in years the majority of them started at the origin, interpreting that the height of a person at birth is 0m. When confronted, the misconception was corrected.

Interestingly in the question on cars A and B a teacher believed that only one of the cars was travelling a straight line as it passed the origin, needless to say that we do not know if the cars were travelling in a straight line or not, only that their speeds were constant.

4.8 Contextual questions as a means of remedying misconceptions

When considering contextual questions, Williams' (2007) believes such questions can connect mathematical concepts to real world conditions relating to everyday life. In examining the contextual question on weight and the difference between boys and girls and which group grows faster, the context detracted from the mathematical ideas as learners' and some teachers' perceived the reality of the context differently to the given mathematical information.

Interestingly two teachers contradicted themselves because of their perceived realties of growth rate of boys and girls at age 14. One of the teachers' stated during the interview that the information (context) would mislead the learners as 'they think that boys grow faster than girls in weight', which is the correct answer, but she answered the question incorrectly in the diagnostic test. Another teacher felt the context would assist learners in this question, because the learners experience is that 'girls grow faster than boys', which contradicted the information given in the diagnostic test. Julie, et al. (1998) suggest that learners need to have an understanding of the context if it is to be effective in assisting learners to overcome misconceptions. Kerr and Maki (1979) believe that contextual questions are only useful if real life context are appropriate to the application of the mathematics being tested and that the solutions found can be tested in comparison to the real world solution.

5. WHAT IS THE LEVEL OF PEDAGOGICAL CONTENT KNOWLEDGE HELD BY THE TEACHERS?

Shulman (1987) believes subject knowledge in itself is not enough to make a difference to learner achievement and that pedagogical content knowledge is necessary to make the difference. If one considers the results in this study, especially those of school 4, where the teacher achieved the top score for the diagnostic test but the learners' class average was one of the lowest, one might feel that they support Shulman's proposition.

5.1 Teacher questionnaires

The questionnaires probed the teachers' level of PCK in three different ways; they had to predict the difficulty level that learners would experience when confronted with the items on the diagnostic test; they then had to identify what errors and common misconceptions their learners would encounter; and then they had to state how they would correct these errors and misconceptions.

5.2.1 Predicting difficulty of diagnostic test items

A five-point difficulty scale varying from very easy to very difficult was provided for teachers' to predict how difficult their learners would find the questions of the diagnostic test. The teachers' predictions of the level of difficulty regularly under-estimated the difficulty of the questions, as the learners always attained below the predicted outcome of the teachers. Hadjidemitriou and Williams (2001) contend that teachers tend to underestimate the difficulty of tasks as they share the misconceptions that their learners hold. Some irregularities were found in how the teachers appraised the difficulty scale to the questions of the diagnostic test, and these

irregularities will be discussed below under the contextual and non-contextual questions of the test.

Contextual Questions

The first and third questions relating to desks and transport showed the best correspondence between learner achievement and teacher expectation. However, despite teachers regarding the transport question very easy to easy. The results revealed that learners performed best in the part (a) of the transport question, but poorly in part (b).

The remaining questions showed a far greater discrepancy between teacher expectation and learner performance. The second question on Mary and John was rated as average by the teachers but only one learner got the answer right. The questions on cars were rated very easy to average yet the learners performed poorly in the question. The charity question was rated easy (with one teacher rating it as difficult) yet less than ten learners obtained the correct answer for (a) and not one of the learners achieved the correct answer for (b). The ratings for the weight question were easy and very easy, yet less than 25 learners correctly answered (a) and only five learners answered (b) correctly.

Non-contextual questions

In the second section of the test, teachers rated the questions as being more difficult. They felt that the learners would find them more complicated and technical than the questions in part A of the diagnostic test. However, once again the teachers underestimated the difficulty that the learners would encounter as they achieved far below the expectations; except in question 2 were the learners' performance and teachers' prediction are similar. Overall, the learners did

perform more poorly in the non-contextual questions with less than 20 learners attaining the correct answers, except in three sub-questions.

5.2.2 Identification of learner errors and misconceptions

Van Dyke and White (2004) suggest that learners they do not know which aspects of the graph to focus on and thus struggle with graphical representations. The teachers in this study also felt that interpretation and reading of graphs was a common problem for learners and could identify aspects of graphs, i.e. intercepts and point of intersection, as areas where possible problems could be encountered. They named rate of change and intercepts as problematic areas, aspects which have been identified as being important by Goerdt (2004).

Jones (2004) believes that teachers with good PCK are able to identify what their learners misunderstand. However, in this study very few teachers could identify common graphical misconceptions found in literature and gave some very superficial ideas when identifying the errors and misconceptions that learners' would encounter, suggesting that the teachers have a low level of PCK.

5.2.3 Suggestions on how teachers could correct errors and misconceptions.

Shulman (1986) sees PCK as the ability to minimise learners' misconceptions and difficulties. An (2004) echoes Shulman where she describes two important roles of PCK as being able to build on students' mathematical ideas and address and correct students misconceptions. In this study many of the interventions suggested by the teachers on how to correct the errors and misconceptions of their learners were superficial and demonstrated a low level of engagement

with respect to the deeper needed pedagogical content knowledge to help learners overcome difficulties and misconceptions.

Many of the suggested methods for correcting the errors and misconceptions included in the statements the words *show*, *explain* and *teach them how*. These words suggest what French (2002) describes as demonstrating 'worked examples' that reinforce procedures through repetition and extensive practice. However, French argues that these exercises do not lead to meaningful and purposeful development of sound algebraic ideas.

The practice of calculating the gradient or slope was also often mentioned as a means of remedying certain errors and misconceptions. Considering Crawford and Williams (2000) study where they found that learners find the calculation of slope easy enough given the equation it did not necessarily indicate understanding the concept of slope and what it represents. If teachers see practising the calculation of slope as an approach to correct an error, what proficiency in learning and understanding of the gradient concepts is taking place? Kilpatrick, Swafford and Findell (2001) define understanding as knowing more than just isolated facts but integrating these into an organised knowledge system which connects new ideas to pre-existing theory.

Another relatively frequent proposal from the teachers for correcting misconceptions and errors in graphs was to plot or draw graphs. However Kerslake's (1981) study found that learners struggle to identify any point on a graph other than the ones they have plotted, ignoring the relationship between the variables.

An interesting teacher comment on the questionnaire was 'Let them learn and memorise the types of slopes and their meaning' as a means of correcting and eliminating errors and misconceptions. This teacher was not concerned with helping them understand what the concept

of slope means, but that simply that the types of slopes and their definitions must be known. The teacher was happy to reinforce through repetition the definitions without paying attention to understanding; which means that the ideas are removed from meaningful and purposeful interpretation of what slope means (French, 2002). The ability to understand according to Kilpatrick, Swafford and Findell, (2001) enables proficiency in mathematical learning.

6. THE NATURE OF THE RELATIONSHIPS BETWEEN TEACHER'S LEVEL OF CONTENT KNOWLEDGE, PCK AND MISCONCEPTIONS ABOUT GRAPHS AND THEIR LEARNER'S ACHIEVEMENT

Da Ponte and Chapman (2006) believe that SCK is a critical component of successful mathematics teaching. Taylor and Vinjevold (1999) found in their study that teachers displayed a lack of content knowledge pertaining to graphicacy. Many other studies pertaining to specific mathematical topics have also found that teachers have SCK deficiencies, which are often mirrored by learners (Even, 1990; Hershkowtiz & Vinner, 1984; Linchevsky & Vinner, 1989; Mastorides & Zachariades, 2004; Pinto & Tall, 1996; Ponte, 1985; Shriki & David 2001; Thomas, 2003; Tirosh, Graeber & Glover, 1986). Many of the teachers in this study held the same misconceptions found in literature that were evident in their learners' understandings.

The teachers' in this study showed a deficiency in PCK as many lacked imagination in identifying possible misconceptions and gave superficial suggestions on how to intervene to eliminate the problems encountered by their learners. Ball and McDiarmid (1990) believe that PCK, the knowledge of how to teach mathematics, rests on the level of SCK held by the teacher. Irwin and Britt (1999) concur when they suggest that teachers that have good SCK are more likely to show more flexibility in their teaching approaches and exhibit greater PCK.

According to Shulman (1986) individual teachers PCK develops differently and it is the quality of PCK that is important. Rossouw and Smith (1998) found in their study that after training in-service teachers had developed vastly different levels of PCK, mostly shaped by their own experiences and perceptions. The teachers in this study had been teaching mathematics for between 8 and 29 years, indicating vast amounts of teaching experience. One would expect the teachers' to display a high level of PCK considering their exposure to the classroom environment. They had small classes, the number of learners taking higher grade ranged between 9 and 20 learners per school, which should allow them to interact closely with individual learners, and to learn from these experiences over the years. However, it appears that experience is not enough to develop PCK; and that consideration needs to be given to what experiences these teachers had when developing their SCK. Ball and McDiarmid (1990) highlight that if teachers do not have sufficient SCK they are in danger of teaching inaccurately and reinforce misconceptions. However, these teachers had just experienced 18 months of direct mathematics instruction at first and second year university level.

This study also showed a poor performance by learners in the diagnostic test indicating that leaner knowledge was far below what could be expected of them according to their age. Even if one looks at the results of School 4 (figure 4.5) where the teacher displays a relatively high SCK the learners still performed poorly, suggesting that the teacher has a low level of PCK. Schön (1983) includes the ability to identify and solve problems in order to build learner knowledge as part of the definition for PCK. The low performance of the learners, particularly in School 4, suggests that without sound teacher SCK, PCK, and understandings of learner misconceptions in graphicacy, learner achievement will remain meagre.

7. SUMMARY

This chapter discusses the findings recorded in chapter four in an attempt to answer the research questions posed in chapter 1. The data on the level of subject content knowledge held by the teachers suggests that their understanding of graphs was not sufficiently developed. The data also revealed that the level of content knowledge held by the teachers was better that that held of the learners, but not exceptionally so.

The common misconceptions relating to graphicacy that were held by the teachers and learners alike were discussed against the findings of other studies while the perceived level of pedagogical content knowledge held by the teachers was discussed with in terms of their predictions as to what they believed learners' would find difficult, identification of errors and misconception, and approaches to remedy these problems.

The chapter concluded by examining the main question of the study, viz. do relationships exists between teacher's level of content knowledge, their PCK, their misconceptions about functions and graphs and their learner's achievement in terms of using graphs and, if there are, what is the nature of these relationships? The nature of the relationships between the teachers SCK, their PCK, their knowledge of commonly held misconceptions, and how these issues affect learners' achievement, are discussed.

CHAPTER SIX

CONCLUSIONS AND RECOMMENDATIONS

1. INTRODUCTION

This chapter provides a summary of the main findings of the study, the conclusions that can be drawn with respect to the relationship between SCK, PCK and learner achievement, considers some of the limitations of the study, raises questions that have arisen and makes recommendations for possible further research in relation to SCK, PCK and learner achievement in mathematic. Firstly, evidence from the diagnostic tests, teacher interviews and questionnaires support the notion that there was a low level of SCK knowledge held by the teacher. The PCK held by the teachers was judged to be superficial as their perception of their learners' difficulty was inconsistent with what the learners themselves experienced as difficult, due to underestimating technical difficulties or overestimating the simplicity of questions. The questionnaire responses also suggested that the teachers' PCK is low as they showed no effective intervention strategies to address problems and issues which may arise in the classroom. The teachers' ability to comprehend their learner's misunderstanding of concepts was also limited, partly because the teacher showed little awareness of common misconceptions themselves, or because they harboured the same misconceptions. The evidence from the learner diagnostic test showed the learners SCK was superficial and mirrored the teachers' responses; they

demonstrated a very low level of achievement and displayed many common misconceptions in terms of graphicacy.

2. IMPLICATIONS OF THIS STUDY

As already discussed, teachers with average to poor SCK knowledge are unlikely to hold significant PCK and their learners are likely to mirror their perceptions and misconceptions. Learners perform poorly, partly because the teachers are under the same misapprehensions as they are. Although these issues have been highlighted in the literature, the results of this study contribute to the academic debate raised by a number of researchers (Ball, Hill & Bass, 2005; Ball & McDairmid, 1990; Da Ponte & Chapman, 2006; Hill, Shilling & Ball, 2004; Leinhardt, 1990; Stein, Baxter & Ebert, 1993), and place the issue in an African context, in terms of what teachers need to know and what should be dealt with in teacher training. It also contributes to the quest for consensus in terms of identifying what mathematics content knowledge is required for effective teaching, what role this SCK plays in developing of PCK, and ultimately how to get learners to perform successfully.

In terms of graphicacy in particular, Leinhardt, Zaslavsky and Stein (1990) reviewed literature on graphs and functions and found that very few looked at the teaching of these topics. This implies that new ideas and approaches are required to build teachers' SCK and PCK in terms of graphicacy, and to make explicit to teachers their learners (and their own) misconceptions in the area. The data also suggest that much work is needed to be done in terms of current measures used to improve PCK and SCK as the 'standard' ways of approaching the problem (as were used in this study) appear to be less than optimally effective. The findings also suggest that there is fertile ground in this area for curriculum developers and trainers of in- and

pre-service teachers, and that effective efforts to improve teachers understandings on the topic need to be researched more deeply.

3. RECOMMENDATIONS FOR FURTHER RESEARCH

Although the SCK of the teachers in this study was below average, the results shed little light on the common assumption that teachers who hold better SCK will have better PCK than those who have less than desirable SCK, or whether their learners will do better than those who have teachers with limited SCK. What the study does reveal, is where teachers and learners have problems in understanding graphical issues, and the common misconceptions that they share. These findings should be of importance when attempting to investigate more deeply the relationship between teachers SCK, PCK, and how this knowledge relates to learner achievement; areas that are ripe for more rigorous research. Another question that begs for attention is that of 'If teachers correctly identify their learners' difficulties, does this necessarily mean that they can diagnose or address them?'

This study focused on SCK and PCK held with respect to graphicacy and further research could consider the same issues in terms of other mathematical topics and how these types of knowledge affect learner achievement in these contexts. An exploration of different settings might provide different information, for example compare learners in highly functional schools, less functional township and rural schools in an attempt to probe whether this is a truly universal problem as suggested in literature or unique to particular settings such as the rural milieu of the Keiskammahoek region. It would also be interesting to look at the relationship between SCK and PCK and learner achievement where teachers training qualifications were similar and different to those in this study. Another possibility would be to extend the research beyond the mathematical

classroom and into other learning areas and see if there is a relationship between SCK, PCK and learner achievement in other subjects.

4. LIMITATIONS OF THIS STUDY

The study was a snapshot of what happened within the localised area of Keiskammahoek and cannot be generalised to broader contexts, as using a non-probability convenience sample (i.e., selected on the basis of availability) is not representative of other groups (Struwig & Stead, 2001). Nevertheless, although the results show that the teachers in this study held a fairly low level of SCK and PCK with respect to graphicacy. Generalisation in terms of mathematics and teachers in general is not applicable because both the issues of sampling and focus on graphical issues. However, according to Fein and Hillcoat (1996), the general conclusions could still be of value as a starting point when investigating the same or similar problems similar situations.

Another limitation of the study is that it did not take the dedication levels and emotional factors of the teachers into account; one of the facets that Ball and McDiarmid (1990) see as important. Also, although the study investigated the relationship between teachers' SCK, PCK and learner achievement, it did not look at finding ways to inform teachers about their learners' misconceptions and the common graphical misconceptions that exist for more effective teaching methods. Nevertheless, as stated earlier, the findings have value in terms of informing the debate around what mathematical knowledge teacher's need, what is considered as important pedagogical content knowledge and what common misconceptions play a role in teaching and learning graphicay, particularly within a South African context. The findings hint at technical issues of language and the teaching and learning of mathematics, but not at the burning issue of language of instruction in South African schools where both the teachers and the learners operate

in a second language. An explicit focus on language use, home language learning, and code-switching, would most likely provide new insights in terms of teacher and learner understandings of the mathematics they are expected to teach and learn, and the complex relationships between teacher's subject content knowledge and pedagogical-content knowledge, and what their children learn in their classes.

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APPENDICES

APPENDIX A

Pupils' Questionnaire CA

GRAPHS

Surname _____

Name _____

School_____

Year_____

Date_____

This is not a test and you will not be marked.

Read carefully and solve the following exercises.

Please write what you are thinking while answering the questions.

If you do not explain your answer (when required) it is considered incomplete and not marked as correct.

When you have finished, please circle the Part you found easiest:

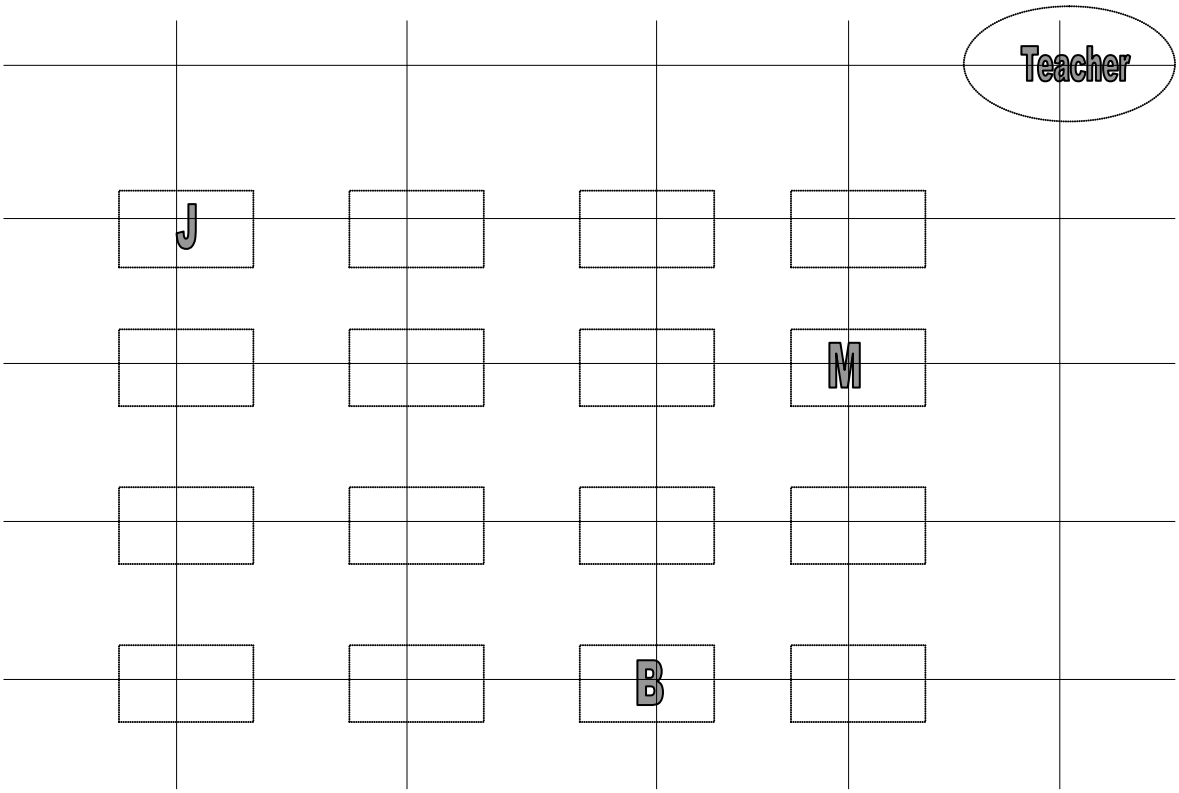
Part One Part Two Both None

Part One.

1. A teacher has given coordinates for each desk in his class. Maria is marked with an ‘M’ on the diagram below. Her desk has coordinates (1,2). John is marked with a ‘J’ and his coordinates are (4,1).

Brian’s desk is marked with a ‘B’.

(a) Write down the coordinates of Brian’s desk: (,)

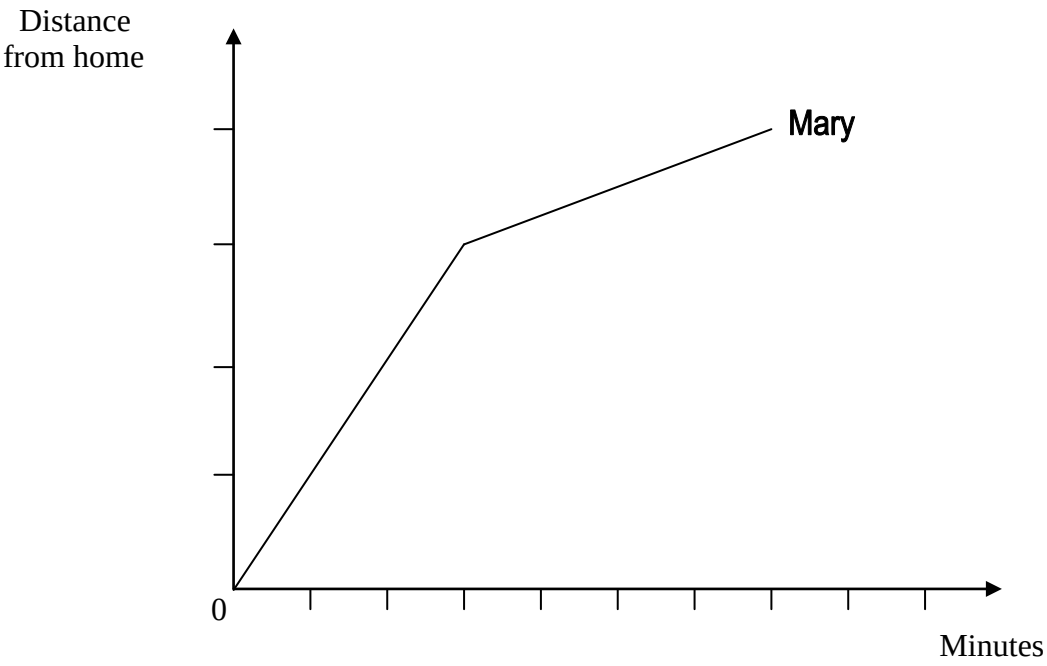


(b) Shade in all the desks whose coordinates add up to 4.

2. John and his sister Mary take 35 minutes to go to the cinema.
Mary runs for 15 minutes then walks for 20 minutes.
John walks for the first 20 minutes then runs for 15 minutes.
Mary runs at the same speed as John.
Mary walks at the same speed as John.

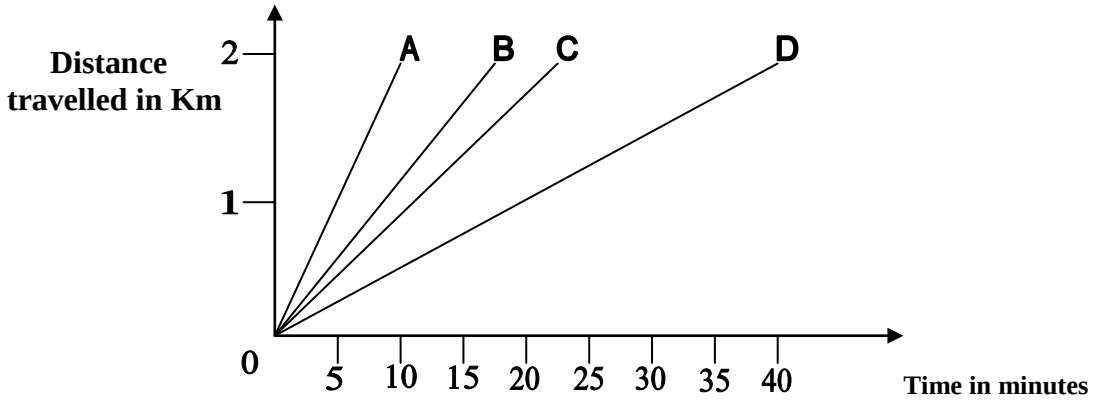
Sketch the graph showing John's journey on the axes below.

Mary's and John's journey to the cinema



3. The graph shows journeys by four different means of transport from home to school, a distance of two kilometers: Bus, Car, Walking, Bicycle.

Distance travelled from home

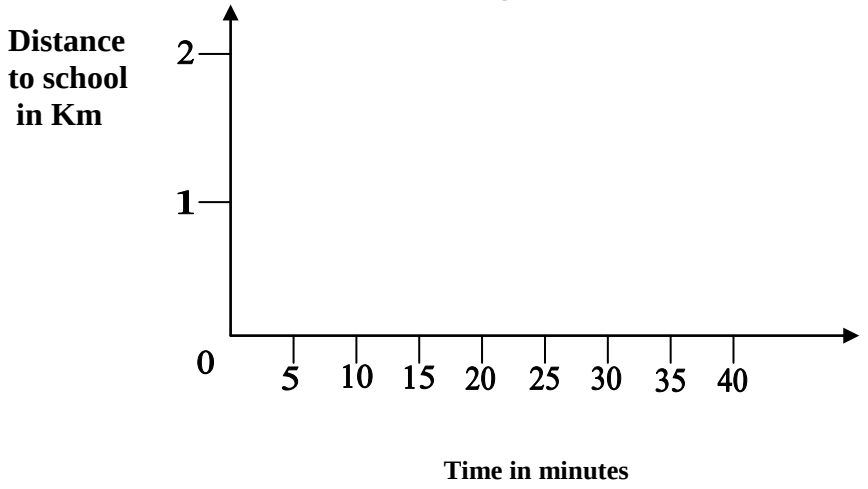


(a) Match each line with the appropriate transport. One has been done for you.

A	Bus
B	Car
C	***Walking***
D	Bicycle

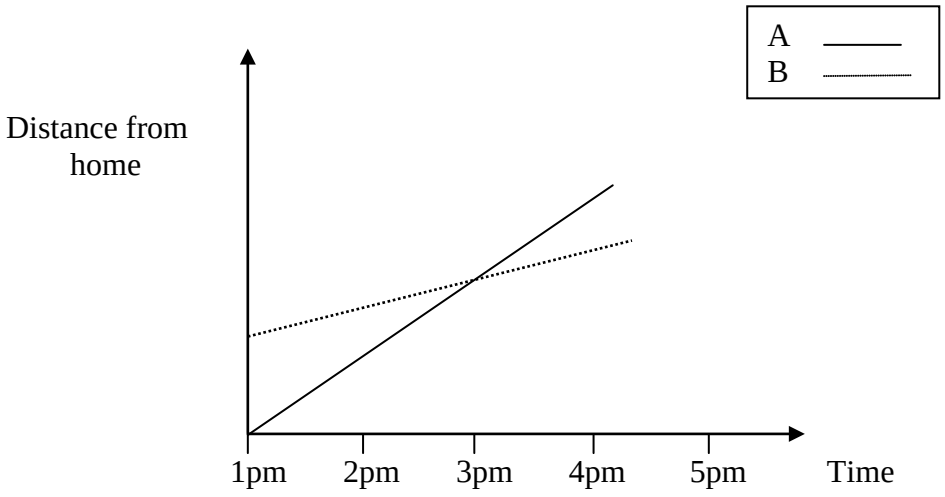
(b) You are going from home to school by bicycle (line C above). Draw a graph below to show how far you have left to get to school according to time.

Distance left to get to School



4. Look at the graph below.

The Distance of Cars A and B from home.



(a) What is happening to the two cars at 3 o'clock?_____

_____

_____

Julie looks at the graph and says, 'At 2pm, the speed of car B is greater than the speed of car A.

(b) Do you agree with Julie? Circle YES or NO.

Give your reasons _____

_____

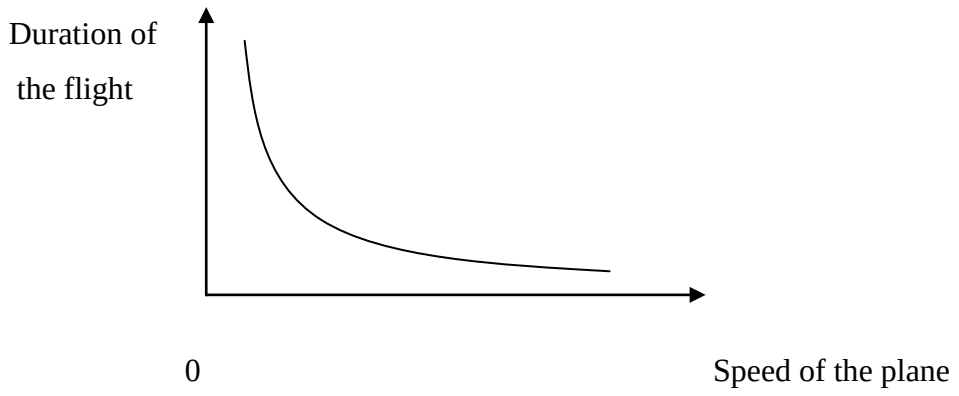
_____

_____

5. For each graph:
Write in one or two sentences interpretation of what the graph is saying. Draw the last graph yourself and explain it.

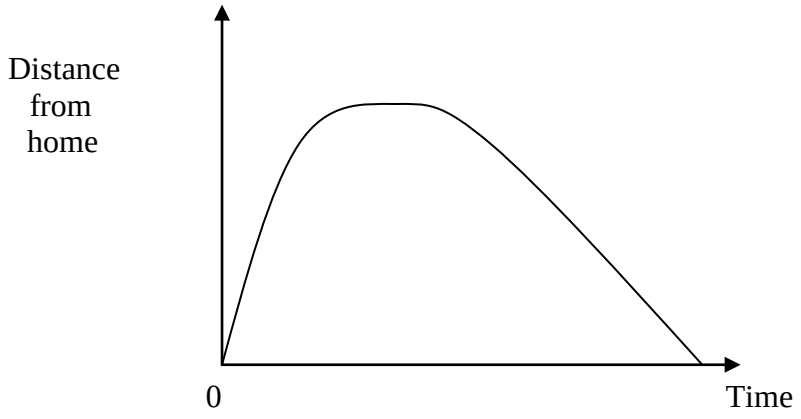
An example is given for you.

A flight to France

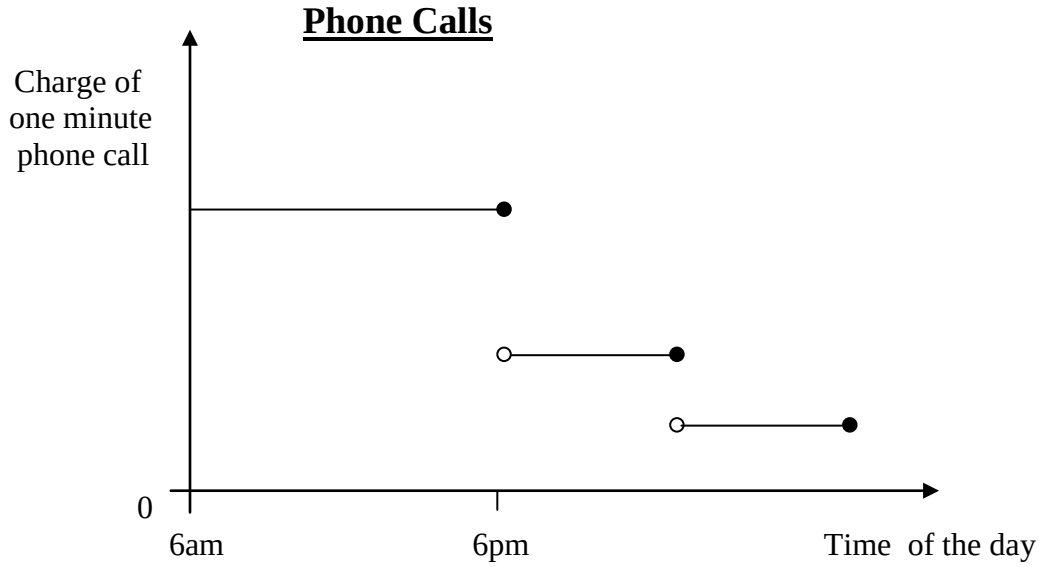


The higher the speed of the plane, the shorter the flight will be. If the speed of the plane is small the flight will last longer.

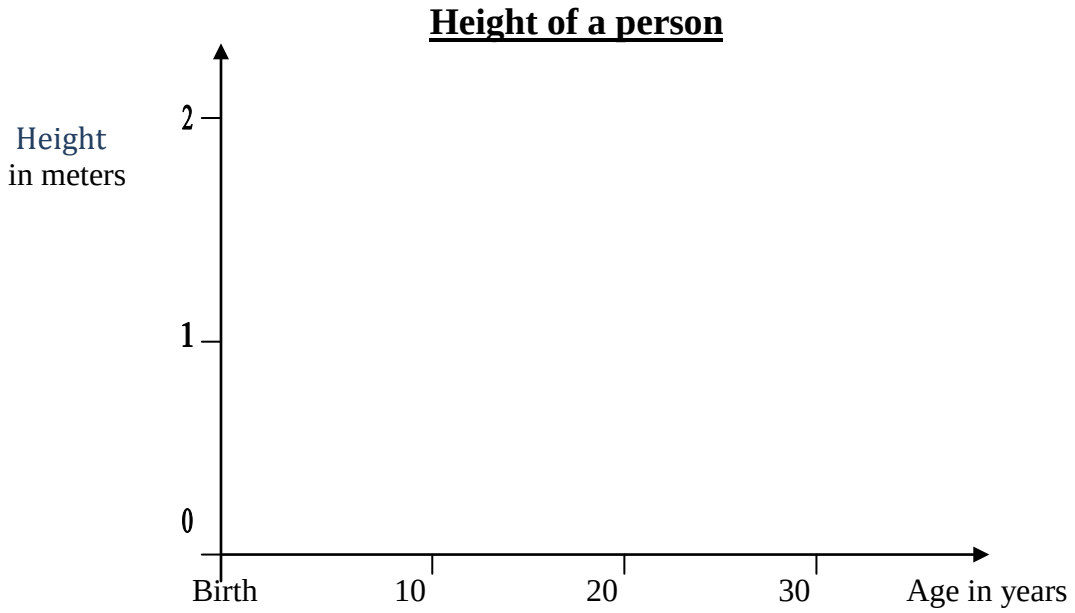
(a) **Peter's Walk**



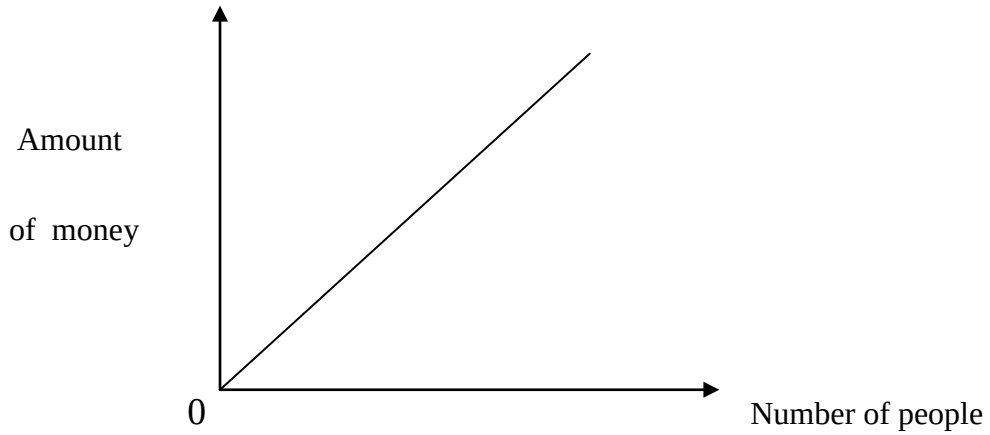
(b)



(c) Draw this graph yourself and then write in one or two sentences interpretation of what it is saying.



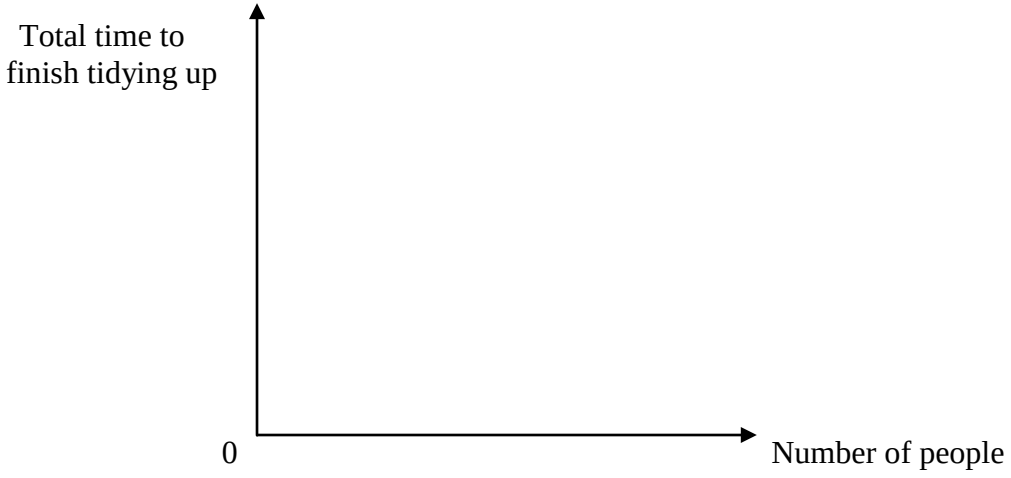
6. At a charity concert each person who comes pays an entrance fee.
(a) Someone says, ‘The more people come, the more money we will collect’.



Explain why the graph goes through the origin in this situation _____
_____

_____

(b) Someone else says, ‘The more people help, the sooner we will finish tidying up’.
Draw a line below to show this.

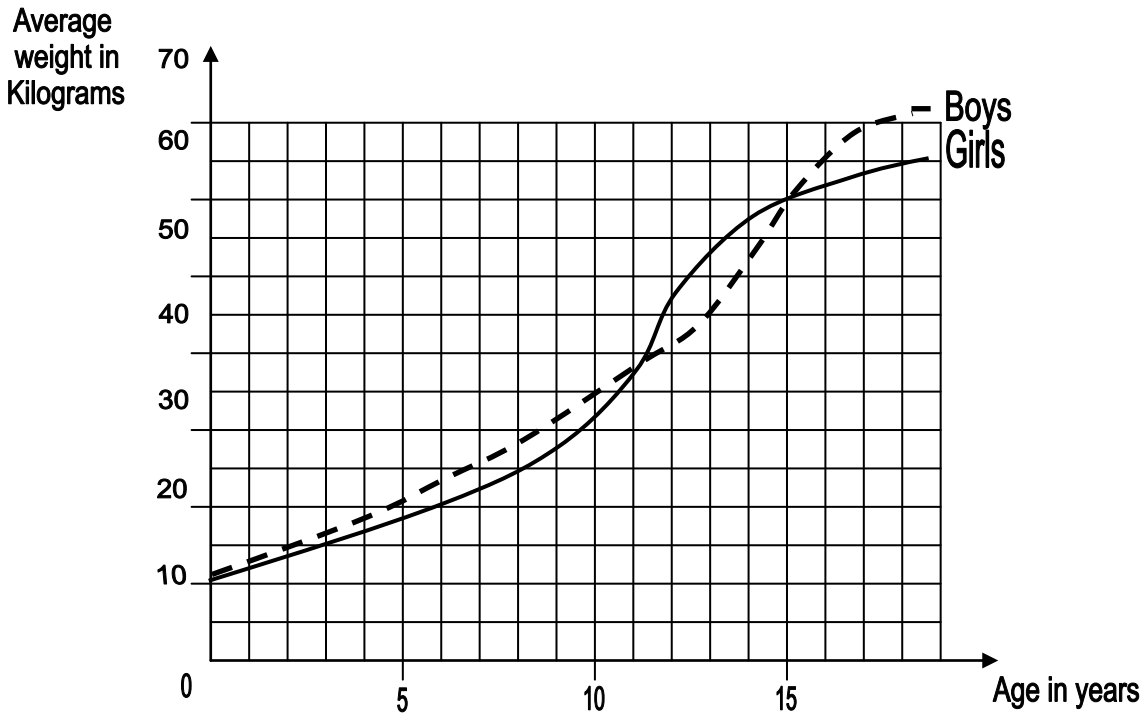


7. Look at the graph below. It shows the average weight of boys and girls as they grow.

(a) What is the difference between boys and girls weight at the age of 14?

_____

Average weight of Boys and Girls



(b) Which group is growing faster at the age of 14? _____

Explain your answer_____

_____

_____

_____

Part Two.

1. John plots the multiples of 4.
The first multiple of four is 4, so he plots (1,4).
The second multiple of four is 8, so he plots (2,8).

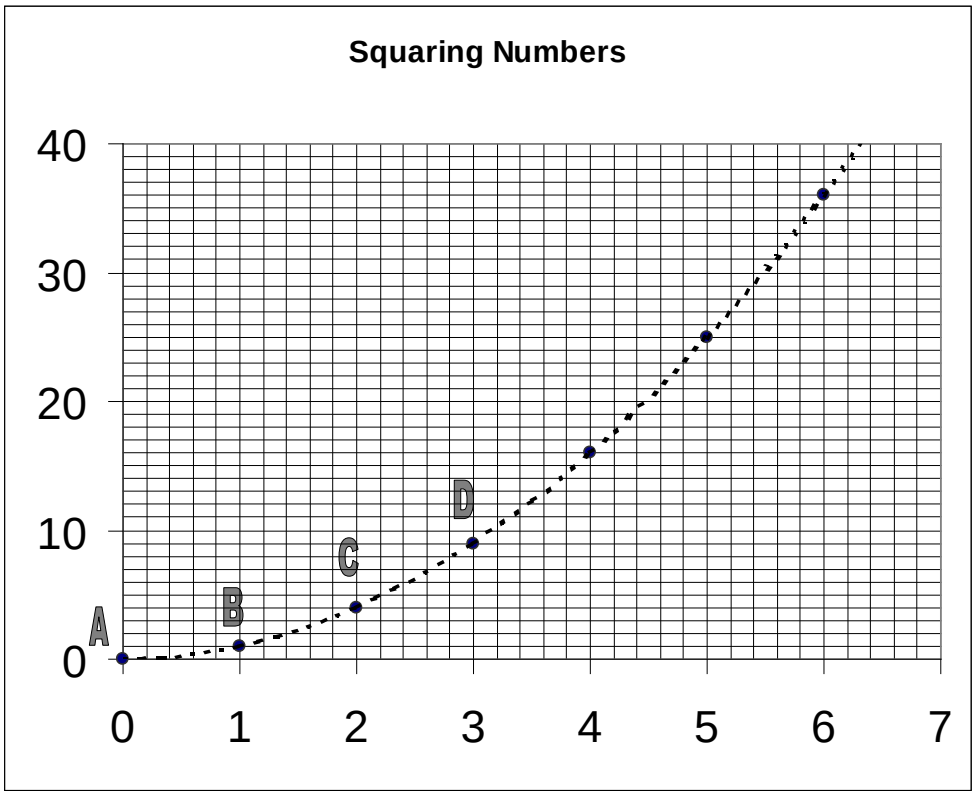
(a) Write down the coordinates of the Point A (___,___).

(b) Fill in the missing coordinates: (20,___) and (___,100)

(c) What is the gradient (slope) of the above line?

How did you find it?_____
_____
_____

2. The graph shows the square numbers.
   A(0,0), B(1,1), C(2,4), D(3,9)



From the graph read approximately

(a) the square of 2.6 ☐ ☐

(b) the square root of 6 ☐ ☐

Does the above graph have the same slope everywhere?

Circle YES or NO ☐ ☐

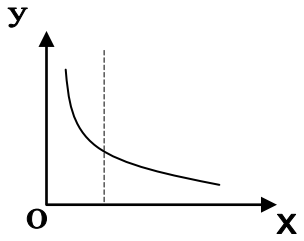
Explain your answer_____

_____

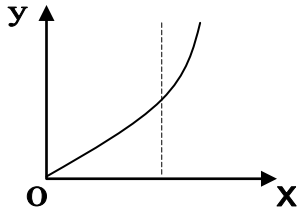
_____

3. Describe in detail the graphs, before and after the dashed line.
An example is given for you.

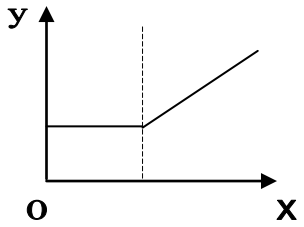
Example:



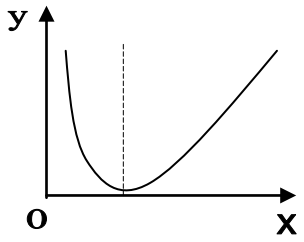
Before the dashed line:
As x increases, y decreases quickly.
After the dashed line:
As x increases, y continues to decrease but slowly.



Before the dashed line:
As x increases, y_____
After the dashed line:
As x increases, y_____

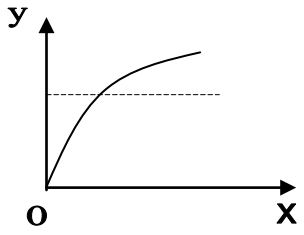


Before the dashed line:
As x increases, y _____
After the dashed line:
As x increases, y_____



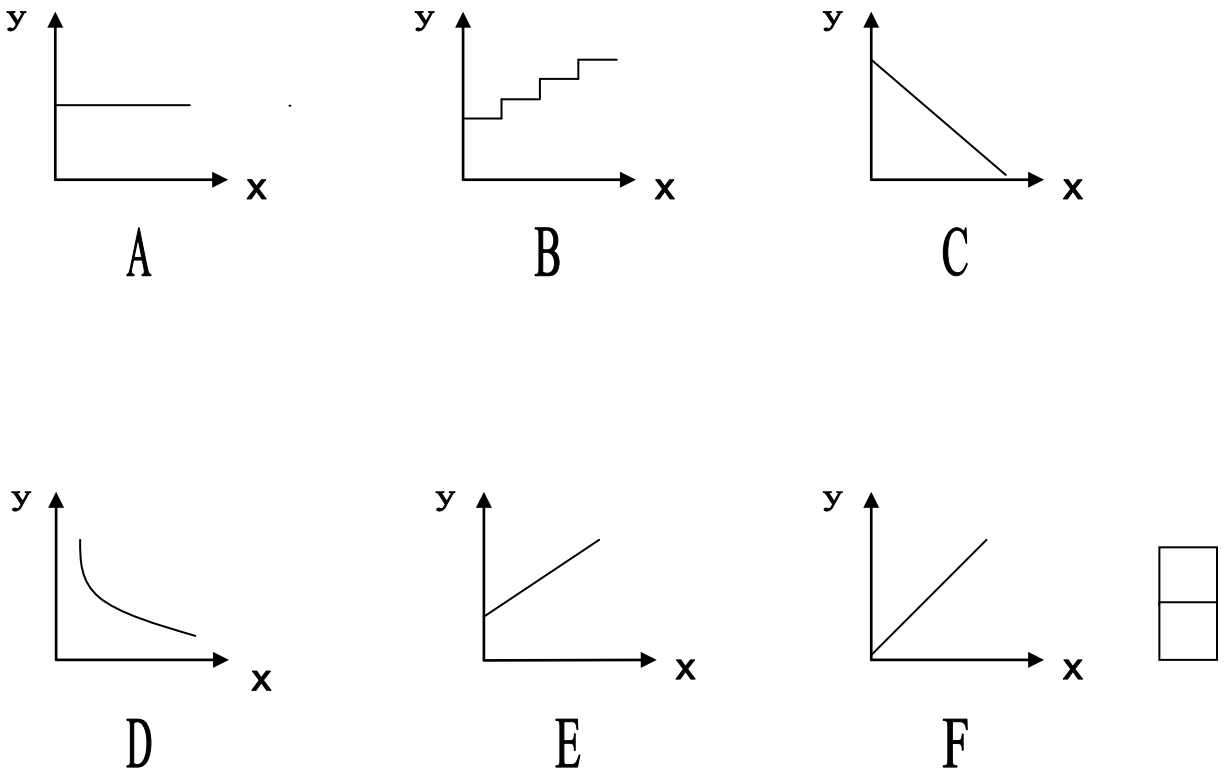
Before the dashed line:
As x increases, y _____
After the dashed line:
As x increases, y_____

NOTICE THAT THE FOLLOWING REFERS TO THE CHANGE OF X

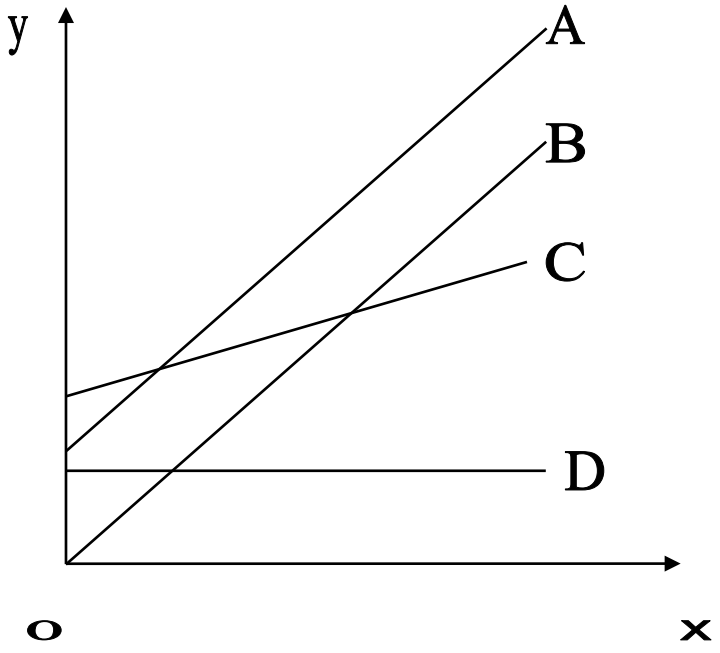


Before the dashed line:
As y increases, x _____
After the dashed line:
As y increases, x_____

4. Which of the following graphs show that y is changing at a constant rate? Circle your answer or answers.



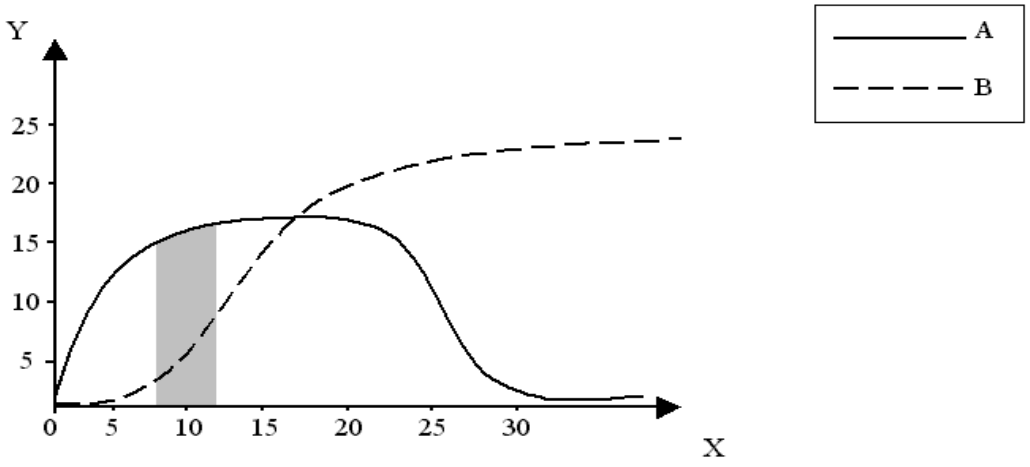
5. Order the straight lines A, B, C and D from least slope to greatest slope.
The lines **A** and **B** are parallel.



Put the letters in order: _____

Explain your answer _____
_____
_____
_____

6. Look at the graph below.



(a) Which of the two curves has the greatest rate of change in the shaded interval? Circle your answer.

A **B**

Explain your answer

_____

_____

(b) At x=25, which of the two curves has the greatest rate of change? Circle your answer.

A B

Explain your answer

_____

_____

APPENDIX B

Teacher Questionnaire on Learners' Graph Test

Surname:_____

Name:_____

School:_____

Date:_____

Number of years teaching Mathematics:_____

Qualifications:_____

Please note that this questionnaire is for research purposes only.
Please answer all three questions as openly and honestly as possible.
Thank-you.

1. Complete the following table by rating how easy or difficult you think that your **learners** will find the various question of the Graph test.

PART 1					
Question	Very Easy	Easy	Average	Difficult	Very Difficult
Desks					
Mary & John					
Transport					
Cars					
Story					
Charity					
Weight					
PART 2					
Question	Very Easy	Easy	Average	Difficult	Very Difficult
1					
2					
3					
4					
5					
6					

2. Suggest errors and misconceptions that you think your learners would be likely to have in terms of each question, i.e. what difficulties do you think your learners will experience in each of the following questions.

PART 1	
Question	Students Errors and Misconceptions
Desks	
Mary & John	
Transport	
Cars	
Story	
Charity	
Weight	

PART 2	
Question	Students Errors and Misconceptions
1	
2	
3	
4	
5	
6	

3. As their teacher suggest how your you could correct the errors and misconceptions they have and help them overcome the difficulties you perceived for each question.

PART 1	
Question	How could you correct the errors and misconceptions
Desks	
Mary & John	
Transport	
Cars	
Story	
Charity	
Weight	

PART 2	
Question	How could you correct the errors and misconceptions
1	
2	
3	
4	
5	
6	