EXPLORING VISUAL PROBABILITY TEACHING STRATEGIES FOR ENHANCING MATHEMATICAL THINKING IN GRADE 11 CLASSROOMS

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ABSTRACT

This Namibian case study aimed to explore the use of visualisation tools associated with different teaching strategies in the teaching of probability concepts in Grade 11 by selected teachers, to promote mathematical thinking. This research project is an integral component of the VIPROMaths project whose goal is to research the effective use of visualisation strategies in the mathematics classroom in the Southern African region. As a mathematics teacher, I have observed that mathematics teaching practices in our classrooms have relatively little connection with actual mathematics and as a result, teaching misses opportunities to promote mathematical thinking. This qualitative case study is underpinned by an interpretive paradigm and it is informed by the dual coding theory. Data was collected through survey questionnaires, reflective journals, field notes, observation schedules and stimulus-recall interviews.

Firstly, I piloted my study by conducting a survey with the Grade 10-12 mathematics teachers in the Khomas region. The aim of this survey was to understand and explore how teachers in the Khomas region taught probability prior to the intervention programme. The data was analysed quantitatively using descriptive statistics such as tables and bar graphs. The findings from the survey necessitated the need for an intervention programme with some teachers in the region, focused on the use of visual tools to promote mathematical thinking.

Lastly, three schools were selected from which three Grade 11 mathematics teachers were chosen to take part in an intervention programme. The goal was to observe how these three teachers use visual probability teaching strategies to enhance mathematical thinking after participating in an intervention programme. Lesson observations showed that all observed teachers used visual models to generate images and used models to develop a probability idea as well as to create platforms for classroom discussions. Interviews revealed that teachers' views towards probability have shifted from that of being the centre of knowledge to that of a facilitator. As a result, teachers used different models to build on learners' prior knowledge, to assess whether they grasped the probability concept and extend their teaching to real-life situations.

This study concluded that the teachers need to consider using mathematical models for creating a platform for discussion to ensure that their verbal explanations are in line with the visuals incorporated. Coupled with that, the teachers' correct use of visual probability teaching strategies has the potential of enhancing learners' mathematical thinking. Therefore, teachers need to teach the learners how to create visuals for enhancing maximise understanding of probability concepts in mathematics. Furthermore, it is hoped that the findings will be useful to mathematics teachers, scholars and educators to improve the teaching of probability.

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DECLARATION OF ORIGINALITY

I Lavinia Tangi-Jehova Nghidinwa, student number 19N9934 hereby declare that this thesis entitled "Exploring visual probability teaching strategies for enhancing mathematical thinking: A case of selected grade 11 mathematics teachers" is my own work and a product of my research. It has not been submitted in any form to another institution. Where I have drawn on ideas of people from other publications or other sources, I have fully acknowledged these in accordance with Rhodes University, Education Department reference guide.

Signature: L. T Nghidinwa Date: 24.09.2021

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ACRONYMS USED IN THIS STUDY

ACR	Analysing conditions of random events
AM	Addressing misconceptions
AMM	Applying mathematical methods and procedures of probability
CMM	Constructing mathematical models for stochastic situations
DCT	Dual Coding Theory
DMT	Developing Mathematical Thinking
EMMS	Encouraging multiple strategies and models
EVP	External visual presentation
FSM	Focusing on the structure of the mathematics
ICT	Information Communication Technology
IRE	Identify random events
JSC	Junior Secondary Certificate
JUMP	Junior Undiscovered Math Prodigies
MoEAC	Ministry of Education, Arts, and Culture
NCTM	National Council of Teachers of Mathematics
NSSCO	Namibian Senior Secondary Certificate for Ordinary level
PLC	Pressing learners conceptually
SRI	Stimulated-recall interview
TLIS	Taking learners' ideas seriously
VIPROMaths	Visual processes in mathematics education
VNVC	Verbal and nonverbal strategies

CHAPTER 1
CONTEXT OF THE STUDY

1.1 INTRODUCTION

The purpose of this study was to explore how selected mathematics teachers used visuals to teach probability for enhancing mathematical thinking in Grade 11 learners, as a result of an intervention programme. In this introductory chapter, I introduce my study by giving its background and context in which I discuss how this study came into existence and how it relates to the visual teaching of mathematics and probability in particular. I also emphasize how probability is currently taught in the Namibian context with special reference to visualisation in mathematics education. The chapter further touches on the research goals and research questions that guided this study. Furthermore, I summarize the methodology used, the data analysis, the theoretical framework and the rationale of the study. Finally, the chapter concludes with a short outline of the structure of the thesis by providing a preview of what is discussed in each chapter, coupled with some concluding remarks.

Moreover, this study is an integral component of the visualisation processes in mathematics education (VIPROMaths) that have a particular focus on visualisation processes in mathematics education in the Southern African region.

1.2 BACKGROUND AND CONTEXT OF THE STUDY

This study focused on the Grade 11 mathematics teachers from the Khomas region. Thirty-four (34) mathematics teachers from secondary schools in the Khomas region were surveyed to explore how they teach probability to promote mathematical thinking. The Namibian National Curriculum for Basic Education, Arts and Culture compels mathematics teachers to be “creative, innovative, to produce their teaching and learning materials linked to practice” (Ministry of Education, Arts and Culture [MoEAC], 2010, p. 6). Coupled with this, one such practice is the effective use of concrete objects to enhance the teaching of somewhat challenging learning areas such as probability in the classroom. Probability, defined here as a branch of mathematics that deals with calculating the likelihood of given events happening or not (Taylor, 2011), is one of the topics in Namibia’s secondary school mathematics curriculum (Ministry of Education Arts and Culture [MoEAC], 2010). Despite the call from the Ministry of Education for teachers to make use of teaching and learning aids and materials, many teachers still find it difficult to create and make use of these teaching aids and materials. Miranda and Adler (2010), for example, observe that “Namibia is one of the many African countries, in which the use of manipulatives in mathematics classrooms is not a common practice” (p. 17).

Probability presents real-life mathematics and connects main areas of mathematics such as counting, statistics and geometry (Grinstead & Snell, 2012) – hence the need for effective ways of teaching probability. In their study done in South Africa, Makina and Wessels (2009) posit that the introduction of probability in the secondary school curriculum needs a serious reconsideration of the way it is taught to the learners. The teaching of probability, therefore, prompts the need for developing teaching strategies that promote the effective teaching of mathematics in general.

Some studies, such as Taylor (2011) and Jones (2006) have focused only on teachers' understanding of the meaning of probability, while others, such as Dollard (2011), focus on a misunderstanding of basic concepts related to probability. However, only a few studies in Asia, South Africa, and Namibia have focused on learners' understandings of probability, such as Nicolson (2005) and Abisai (2018).

In my career as a mathematics teacher and external examination marker over the years, I observed and experienced how learners struggle with probability concepts during teaching and when answering examination questions. The examiners' reports as from 2011 up until 2016 for both the Junior Secondary Certificate (JSC) and the Namibian Senior Secondary Certificate for Ordinary level (NSSCO) showed that “probability” is the topic of concern which always appears as the area of difficult in both junior and secondary grades. This is shown by the examiners' comments for six consecutive years as shown in Table 1.1 below.

Table 1.1: Examiners' comments relating to probability from 2011 to 2016

MoEAC, JSC examiners report (2011).	Probability is still a problematic syllabus topic for the learners (p. 218).
MoEAC, JSC examiners report (2013).	Learners cannot express their answers as fractions or percentages but tend to give just a whole number like 3 instead of $\frac{3}{20}$ (p. 264).
MoEAC, JSC examiners report, (2016).	Teachers should give more practical exercises in probability etc. since it is proved to be a difficult topic for the learners. (p. 231)

MoEAC, (NSSCO) examiners report (2011).	Few questions on probability were answered moderately, many learners failed to give fractions in their simplest form. (p. 332)
MoEAC, (NSSCO) examiners report (2013)	Most of the questions were poorly answered, for example, questions like use words **certain, impossible, and unlikely** to describe the probability of a **rainstorm in Oshakati on June 25**, **being born on 31 April,** and the probability **that the sun will rise tomorrow**. Some learners gave the numbers instead of using the given words and others gave explanations on the given information which indicated that many learners had no ideas with this questioning on probability. (p. 333)
MoEAC, (NSSCO) examiners report (2016).	Probability seems to be challenging, it is part of the specific syllabus components that learners struggle with. Some learners give their answers in ratio form. Other learners have a problem with a word like "or" and "and" used in the questions. (p. 337)

Additionally, the Ministry of Education Arts and Culture [MoEAC] (2014) reports on the Grade 12 examination (see Table 1.1) indicate that learners studying mathematics on an extended level find it difficult to answer questions on probability in Paper 4. If unaddressed, this situation could compromise the promotion of mathematical thinking in learners, hence there is a need for intervention. Based on the arguments above, the concept of probability has proved to be problematic in teaching processes at both junior and secondary levels. It was the intention of this study, which is structured around an intervention programme to explore the potential of visual probability teaching strategies, to enhance mathematical thinking in Grade 11 classrooms.

1.3 RESEARCH GOALS

The overall goal of this study was to explore how visual probability teaching strategies can be used by the selected mathematics teachers that participated in the intervention programme to teach probability for enhancing mathematical thinking in Grade 11 learners

1.4 RESEARCH QUESTIONS

This study sought answers to the following questions:

1. How did secondary school mathematics teachers in the Khomas region teach probability to promote mathematical thinking before the intervention programme?
2. How do selected teachers use visualisation tools to promote mathematical thinking during the teaching of probability concepts in Grade 11 classrooms, as a result of participating in an intervention programme?
3. What are the selected teachers' experiences of using visualisation tools to teach probability for promoting mathematical thinking, as a result of an intervention programme?

1.5 THEORETICAL FRAMEWORK

The Dual Coding Theory

This study is informed by the dual coding theory (DCT) which is founded on the notion that information for memory is processed and stored by two interconnected systems and sets of code, also known as the verbal and nonverbal systems (Paivio, 2006; Suh & Moyer-Packenham, 2007). The reformed approaches in education suggest that the inter-connected use of verbal and non-verbal codes can enhance the quality of teaching and learning in mathematics classrooms. Visual and verbal codes are interdependent and they complement each other. When used concurrently, visual and verbal codes produce better results as far as teaching and learning are concerned (Stokes, 2002).

Aligning teaching strategies to the DCT can enhance learning in a meaningful way as learners can process knowledge through the two streams, which are the verbal and the visual codes. Rieber (1994) affirms that "it is easier to recall information from visual processing codes than verbal codes because visual information is accessed using synchronous processing, rather than sequential processing." Therefore, the incorporation of the visual processing code is recommended as it enables learners to make connections between related mathematical concepts (and unrelated ones). This study advocates that in a mathematics classroom, teachers should use different visual teaching strategies to teach probability effectively. This could be likened to Paivio's nonverbal code that can enhance the quality of teaching and learning in mathematics classrooms.

1.6 RESEARCH METHODOLOGY

This study employs a qualitative approach to data analysis and is oriented within the interpretive research paradigm. This interpretive paradigm emphasizes that “the view of the world we see around us is the creation of the mind” and “we can only experience it personally through our perceptions which are influenced by our preconceptions, beliefs, and values” (Walliman, 2011, pp. 21–22). Therefore, the interpretive paradigm enabled a rich understanding of situations (planning and teaching). The onus of the research in the interpretive paradigm is to comprehend, articulate and demystify social reality through the eyes of different participants (Cohen, Manion, & Morrison, 2011b).

This study followed a mixed-method approach, involving a case study developed from the perspective of practical activities and the use of visual tools within the classroom. Turner and Ireson (2010) define a mixed-method as a research approach in which quantitative and qualitative data or techniques are mixed in a single research. Data were collected through a survey questionnaire, lesson observations, and stimulated recall interviews as shown below.

Firstly, I piloted the questionnaire with the Grade 10 teachers at my school to make sure the questions were clear to the participants. As a result, the necessary adjustments which included definitions of keywords were made on the questionnaire. During the survey, I targeted all the Grade 10-12 mathematics teachers in the Khomas region. I then distributed the questionnaires through circuit representatives. Thirty-four mathematics teachers responded to the survey questionnaires which were designed to establish how and to what extent teachers were using visuals to teach probability. The questionnaire data helped to craft the intervention programme.

After the survey, there was an intervention programme that involved two workshops with the Grade 11 mathematics teachers. During the induction workshop, I introduced the study to the mathematics teachers and I selected three teachers from three different schools purposively, because they had all taught for more than five years and because their schools were conveniently close to my school. This made travelling to their schools for observation easier. These teachers were willing to take part. We therefore developed the second workshop with the participant teachers whereby we planned around the probability concepts to be taught and the visual tools to be used. These workshops were not video recorded.

I piloted my analytical tools with the Grade 10 class where I was interested checking whether my analytical tools would give me quality data to help me answer the research questions. This lesson was not video-recorded. For confidentiality’s sake, throughout the study, I used pseudonyms instead of the teachers' names to uphold the ethics for conducting a research study at Rhodes University. The intervention programme focused on how to use visuals to construct mathematical models to teach probability, followed by an awareness of promoting mathematical thinking in teaching through using mathematical models. Because of the lockdown of schools due to COVID-19, the intervention took a longer period in three phases. All the teachers started with their first lessons during the second school term of 2020 until the first school term of 2021. Each teacher taught three lessons that were video recorded: one lesson per phase for each teacher.

Just after the lesson observations, I conducted a one-on-one simulated interview with each teacher to establish their views and experiences of how and whether the use of visual tools enhanced mathematical thinking. Each teacher was interviewed once and these interview data were audio recorded and transcribed later.

1.7 DATA ANALYSIS

This study collected both quantitative and qualitative data. The data analysis process underwent three stages. In **Stage 1**, data from the survey questionnaire were first analysed quantitatively using descriptive statistics. The qualitative data from the questionnaire were analysed using themes generated from the questionnaire questions. The participants were given pseudonyms, for example, teacher one as T1, teacher two as T2, and so on.

In **Stage 2**, I quantitatively analysed the data from the lesson observations dually. Firstly, I focused on the visual probability teaching strategies. Here I was more interested in observing how the selected teachers used visual tools to construct mathematical models in teaching probability. These teaching strategies were in conjunction with those of Batanero, Chernoff, Engel, Lee, and Sánchez's (2016) probabilistic reasoning and Paivio's (2006) dual coding theory as seen on the analytical framework A (see Appendix H), as well as the emergent themes from the observed data. Secondly, the focus was on how the mathematical models which were used, promoted mathematical thinking in Grade 11 probability lessons in the context of Carney et al. (2014) observation instrument for developing mathematical thinking – as adapted from the National Council of Teachers of Mathematics (NCTM) - and as seen on the analytical tool B (see Appendix H).

Stage 3 was a thematic analysis approach whereby I analysed the data as per the themes generated from the interview questions. In this last stage, I focused on the teachers' views and experiences of visuals to teach probability concepts.

1.8 RATIONALE OF THE STUDY

Having taught mathematics for twelve years, I have observed that mathematics teaching practices in our classrooms have relatively little connection with actual mathematics and as a result, teaching misses opportunities to promote mathematical thinking. It is hoped that this research will help educators to realize the importance of using visualisation tools in the teaching of probability. This study further hopes that the selected mathematics teachers will be empowered with different approaches involved in using visuals in teaching probability for enhancing mathematical thinking. Although a case study cannot claim generalizability, I believe mathematics scholars and practitioners in Namibia and elsewhere, will find the findings of this study useful and applicable beyond this case study where visualisation tools such as dice, coins, decks of cards, coloured balls and cubes were used in the teaching of probability to enhance mathematical thinking. The study will further help learners to link (make connections) between a concept and real-life maths problems when using visualisation tools in probability.

1.9 OPERATIONAL DEFINITIONS

In this section, I define the key concepts as used in my study.

Probability refers to a study of random phenomenon or the chance or likelihood of an event to occur Xiayan (2015).

Visualisation: The term visualisation is used to refer to both internal mental representations and external displays. Arcavi (2003) defines visualisation as:

The ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools, with the purpose of depicting and communicating information, thinking about and developing unknown ideas and advancing understandings (p. 217)

Internal visualisation: is a representation in the mind of an individual derived from imagery or imagination (Hegarty, 2004).

External Visualisation: is a representation in the environment that can be perceived by an individual (Hegarty, 2004).

Mathematical thinking: Devlin (2011) states that “mathematical thinking is a whole way of looking at things, of stripping them down to their numerical, structural, or logical essentials and of analysing the underlying patterns” (p. 59).

Conceptual understanding: as described by Kilpatrick, Swafford and Findel (2001) is an integrated and functional grasp of mathematical ideas.

1.10 STRUCTURE OF THE STUDY

This thesis is organized into five chapters as follows:

Chapter 1 (Introduction): This chapter provides an overview of the thesis. In this chapter, the context of the study, coupled with the background of the study is firstly discussed. The chapter further sets out the research goals and questions, the theoretical framework, the research methodology, the rationale of the study and finally the structure of the study.

Chapter 2 (Literature review): begins with a discussion of the literature around visualisation in mathematics education and a critique of visualisation teaching strategies. This literature informs and shapes the analysis and interpretations of the findings of this study. Key concepts are probability teaching strategies, manipulatives, games, visualisation, mathematical thinking, and the Dual Coding Theory. Lastly, the principles underlying the Dual Coding Theory were discussed.

Chapter 3 (Methodology): describes the methodology and methods used to collect the data. Here, the research process and methodology that guided this study are discussed. This chapter further provides an account of how the research was designed and carried out. It discusses the orientation, research methods employed, research design, data collection tools and sampling techniques used in this study. It further discusses issues about data analysis, validity, and ethical considerations.

Chapter 4 (Data presentation, Analysis, and Discussions): is primarily dedicated to the qualitative and quantitative analysis of data collected. The chapter presents the data from the following three research instruments: survey questionnaires, observations and stimulated recall interviews. This chapter further analyses and interprets the data to generate the research findings for the three research questions given above. It also discusses the summary of the findings by linking the data analysed to the Chapter 2 literature reviewed. The analytical tools that were used to analyze the data (see Appendix H) are also presented in this chapter.

Chapter 5 (Conclusion and Recommendations): This is the final chapter of this study and it presents a comprehensive summary of the key findings, drawing conclusions from the findings and followed by the rationale of the study. The chapter further presents the limitations and challenges as well as recommendations based on the research findings that were made.

Lastly, the suggestions for further research and personal reflections of this research study conclude the chapter and thesis.

1.11 CONCLUSION

This chapter provides the reader with an overview of the research study to provide a clear understanding and enable easy navigation through the thesis. The next chapter describes the literature relevant to this study as well as the theoretical framework of the study.

CHAPTER 2
LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.1 INTRODUCTION

The focus of this study was to explore the use of visualisation tools in the teaching of probability to promote mathematical thinking. In this chapter, I present the literature relevant to my study in the following order: firstly, I discuss the definitions of probability and the types of probability as defined by various researchers; secondly, I focus on the teaching and learning of probability in general; thirdly, I discuss visualisation definitions and internal and external visual representations; and fourthly, I look at visual probability teaching strategies. I also focus on mathematical thinking – what it is and its indicators that can be observed within the classroom environment. Lastly, I discuss the Dual Coding Theory that informs this study, after which, the chapter concludes.

2.2 PROBABILITY

2.2.1 Definitions of probability

Probability is the extent to which something is likely to happen or be the case. It is the extent to which an event is likely to occur, measured by the ratio of the favourable cases to the whole number of cases possible (Fulton, Mendez, Bastian, Musal & Mendez, 2012). The concept of probability is not new in research (Dollard, 2011; Nicolson, 2005; Xiayan, 2015) thus there are various definitions of probability based on the classical (equal possibilities based on physical symmetry), frequentist (observed frequencies of events), and subjective perspectives (degree of subjective certainty or belief). The trinity of probability interpretations are described as follows:

2.2.1.1 Classical interpretation

The classical interpretation of probability is deemed as 'a priori probability' because of its alignment with the rationalist perspective (Chernoff, 2008). Probabilities are calculated deductively without the need to experiment, for example, if an experiment has ***n*** simple outcomes, this method would assign a probability of $\frac{1}{n}$ to each outcome. This implies that each outcome is assumed to have an equal probability of occurrence (Chernoff, 2008). Similarly, tossing a coin would be $\frac{1}{2}$ (head or tail), rolling a dice $\frac{1}{6}$, or two rolls of a dice which are independent events, $\frac{1}{6}$ **x** $\boldsymbol{\frac{1}{6} = \frac{1}{36}}$

2.2.1.2 Frequentist interpretation

As a result of the technology available, the frequentist view is to introduce probability as the limit of relative frequencies in a long series of trials (Henry, 2010). This change also shifts from a formula-based approach to an emphasis on providing the probabilistic experience. Therefore, learners are encouraged to perform random experiments (like rolling a dice 100 times or taking records for the daily sales of desktop computers in the past 30 days) or simulations, or formulate questions or predictions about the tendency of outcomes in a series of these experiments. Concurrently, learners are encouraged to collect and analyse data to test their conjectures and justify their conclusions based on these data (Henry, 2010).

Many other curricula in Europe, Australia, New Zealand, and the United States approach probability and inferences in a frequentist way by making use of simulation and resampling to estimate the probabilities of interest (Jackman, 2000).

2.2.1.3 Subjective interpretation

> Subjective probability describes probability as a degree of belief, based upon personal judgment and information about an outcome (Mooney, Langrall, & Hertel, 2014). As a result, it is no longer assumed that "all national human beings with the same evidence will have the same degree of belief in a hypothesis" (Gillies, 2012, p.1).

Moreover, the subjective view that considers once-off frequent decisions in everyday life, and where the frequentist view cannot be applied, it is hardly considered in the curriculum (Borovcnik & Kapadia, 2009a). Moreover, the experiments we often simulate are typical examples of random situations in the sense that in a few real-life applications of probability we can repeat a process many times in the same conditions. Subjective situations (like: should the teacher ask me next time?) where only personal probabilities can be applied, can complete the field of application of probability. Nevertheless, in the context of decision making such as getting an insurance policy it is useful to introduce subjective views (Borovcnik, 2006). Therefore, when heading the uncertainty of a single decision, such a decision could be made clearer if students are asked to weigh up the various possibilities and calculate the expected values of costs or prizes (Borovcnik, 2006).

2.2.1.4 The probability terminologies

Eichler and Vogel (2014) propose a modelling approach for each of the main views of probability (classical, frequentist, and subjective) and discuss the role of simulation in supporting learners' understanding in each of these perspectives. It is crucial for learners to develop a meaningful understanding of probability from all three perspectives. Learners should acknowledge these interpretations and explore connections between them and the different contexts in which one or the other may be useful (Dollard, 2011). Having said that, Devlin (2014) states that, “Probability is the only reliable means we have to predict and plan for the future, it plays a huge role in our lives, so we cannot ignore it, and we must teach it to all future citizens” (p. ix). Most definitions presented above seem to imply that probability involves a study of rules of random phenomena or chance or likelihood of an event to occur (Xiayan, 2015).

Furthermore, Xiayan indicates that probability contains significant mathematical terminologies with specific meanings such as ‘random phenomenon’, ‘fundamental events’, and ‘equal possibility’. These concepts are specific to understanding probability. Moliner (2000) defines “random as uncertain, it is said of what depends on luck or chance” (p.1283). Similarly, Moliner (2000) further defines chance as “the presumed cause of events that are neither explained by natural necessity nor by human or divine intervention” (p. 320). Therefore, one can infer that random is something with unknown causes, and chance is assumed to cause a random phenomenon. It is a model we administer to some situations to predict or control the situations. In contrast, some philosophers relate chance to casualty Bennett (1999).

Democritus suggested that everything is the combined fruit of chance and need, while Leucippus believed that nothing happens at random and everything happens for a reason and out of necessity. Aristotle considered that chance results from the coincidence of several independent events whose interactions result in an unexpected result, and randomness is the measure of one’s ignorance. Therefore, this seems to imply that every phenomenon has a cause. In this study, I am going to use (Xiayan, 2015)’s definition. In this study, probability refers to a study of random phenomena, chance or likelihood of an event to occur. Probability is taught and studied in school, college and university. In the next section, the study looks at the teaching of probability in general.

2.2.2 Teaching of probability in general

French mathematicians first used probability theory in the seventeenth century in games of chance. Since then, probability influenced early research until it became a well- established branch of mathematics that has applications in every area of scholarly activity in daily experience (Grinstead & Snell, 2012).

The understanding of probability is vital in understanding politics, weather reports, genetics, sports, and insurance policies. Thus, learning about probability may enhance learners’ problem-solving skills (Taylor, 2011). Literature Xiayan (2015) notes that probability is an important part of secondary school mathematics that provides methods for a problem-solving and thinking model for people to understand the objective world. It also offers the foundation

Grinstead and Snell (2012) theory for the development of statistics. Thus, probability provides secondary school learners with a good platform for knowing the applicability of mathematics and lays the foundation for their future study. In high school, teachers are likely to use random chance devices like dice and spinners when teaching probability, and describe probability in terms of ‘equally likely outcomes’ and as ‘the ratio of the desired outcome to total possibility outcome’.

Literature has revealed that the development of learners’ mathematical reasoning through the study of probability is essential in daily life, as probability offers the fundamental theory for the development of statistics and problem-solving in science and mathematics-related fields (Nicholson, 2005; Taylor, 2011). Studies by Nicholson (2005) and Taylor (2011) further emphasise the importance of teaching probability. They note that probability plays a major role in modern society both in the daily lives of the public at large and in professionals’ activities within society. Thus, probability calls upon many mathematical ideas and skills developed in other school subjects like sets, mapping, numbers, counting and graphs, and enables learners to work in branches of mathematics which are relevant to current life situations. Presently, in high school, the way probability is taught can be likened to a situation where one is learning to drive a car. In contrast, thinking mathematically can be likened to a scenario where one is learning how a car works, how to maintain and repair it and, or one can go as far as on how to design and build up his/her car (Devlin, 2012).

Taylor (2011) indicates that many high school learners find it difficult to understand probability because of inadequate pre-requisite mathematical skills, abstract reasoning, and a lack of instruction that enables learners to be actively engaged in learning. Consequently, learners may discover and construct their understanding of probability concepts regardless of the factors that contribute to the perceived difficulty of probability.

Nicholson (2005) and Taylor (2011) indicate that the development of learners’ mathematical reasoning through the study of probability is essential in daily life as probability offers the fundamental theory for the development of statistics and problem-solving in science and mathematics-related fields. Probability presents real-life mathematics and connects main areas of mathematics such as counting, statistics, and geometry. Probability is also used in medicine such as in predicting the risk of new medical treatments (Grinstead & Snell, 2012). This indicates that educators need effective ways of teaching probability, thus the need for this study.

2.2.2.1 Teaching probability using games

When considering the use of games for teaching mathematics, educators should distinguish between an activity and a game. Gough (1999) states that a ‘game’ needs to have two or more players who take turns, each competing to achieve a winning situation of some kind, each able to exercise some choice about how to move at any time through the playing. Using games to teach probability may generate excitement, making probability a fun topic for learners. Notably, a game is seen as a subset of both play and fun (Prensky, 2001). It is therefore recognized as an organized play that gives enjoyment and pleasure. Games can be either physical or video. However, amongst all video games, Griffiths (1996) posits that weird games are those that do not fit into the other categories he mentions, so they are not sports simulations, racers, adventures, platformers, platform blasters, beat ‘em ups and shoot ‘em ups. Of the seven types of games that Griffiths (996) identified he asserted that only the puzzlers (brain-teasers) and what he calls ‘weird’ games that contain educational components and can be used in schools. He further alluded to the notion that these video games foster learning and overcome negative stereotypes that many people have about video games. Dempsey, Lucassen, Haynes, and Casey (1996) further define a game as:

… a set of activities involving one or more players. It has goals, constraints, payoffs, and consequences. A game is rule-guided and artificial in some respects. Finally, a game involves some aspect of competition, even if that competition is with oneself. (p. 2)

Moreover, the term ‘educational game’ is used to refer to games designed explicitly to achieve specific curricular goals in school subjects like mathematics and science (Devlin, 2003). In support of this, Liu (2011) suggests that teachers need to use multimedia and provide rich real-life situations and games to facilitate learners’ understanding of probability. Similarly, Xiayan (2015) and Nicolson (2005) outline the significance of using games in improving understanding of probability. Thus, the use of games could be an effective tool for facilitating learning because games motivate, challenge, increase curiosity, and promote fantasy in children (Ferguson, 2014). Therefore, introducing situations such as games in the mathematics classroom can arouse interest in learning and thus deepen learners’ understanding of probability. Such games may also help learners to acquire an understanding of mathematical principles in probability learning and gain basic knowledge, develop logical thinking and acquire skills of recognising, describing and solving real problems by probability methods. Some instructional hands-on games may be used to understand randomness and chance of events such as tossing coins, rolling dice, drawing from candy bags, and spinning spinners as they are quick and basic experiments to do in probability lessons.

While introducing games, teachers should prepare to offer small variations on the standard experiments to reinitiate learners' interest and thinking learners may become bored with the same games. This is also supported by the revised curriculum that states that mathematics teachers need to be "creative, innovative, to produce their teaching and learning materials linked to practice" (Ministry of Education Arts and Culture [MoEAC], 2010, p. 6). Hence, introducing situations such as games in the mathematical classroom can arouse learners' interest in learning and thus deepen their understanding of probability.

However, Dollard (2011) views probability as a difficult topic in mathematics, so there is a need for instruction to build on learners' existing notions of probability. He emphasizes that teachers need to produce probabilistic tasks that are linked to the teaching objectives (Jones, 1996). Moreover, Jones (1996) and Nicolson (2005) note that games are important tools in supporting instruction as they create realistic simulations as well as enable teachers and learners to deal with real-life probability problems. Therefore, involving learners in a series of games on probability such as tossing coins, drawing from candy bags, rolling dice, spinning spinners and many more, may help learners to develop ideas about randomness and chance.

Several studies (Dunn, 2005; Nicolson, 2005) explain that dice are used to determine and understand the probability of simple events, assuming equally likely outcomes. Learners are allowed to roll the dice several times, record the number of times each number (1 to 6) comes up and discuss the results. Tossing coins involves throwing a coin in the air, the coin turns several times in the air and lands randomly on 'heads or tails'. This is done to help to seek and find explanations and interpretations of equally likely outcomes. Drawing candy from the candy bag is used to demonstrate the chances of pulling out candy depending on the numbers of their particular types in the bag, compared to other candy types.

Spinners may be used as a common tool for exploring and understanding classic probability. For each spinner, learners use a circle divided into six equal parts and a paper clip twirled around the point of a pencil. They repeatedly spin and shade the areas where it stops. These games are done to enable learners to predict the next possible outcomes. Hands-on games may be used to create a random, equally likely outcome for experiments in probability and form the connection between mathematics and real-life situations, (Nicolson, 2005).

Liu (2011) indicates that underachievement in mathematics is an ongoing worldwide concern. He pointed out that learners begin elementary mathematics lacking motivation, which continues to secondary school level, which yields poor performance. Part of the reasons may be due to poor attitudes toward mathematics and poor teaching strategies in mathematics.

Therefore, to help remedy poor motivation and increase learner achievement, teachers need to be aware of and implement best teaching practices in mathematics instruction. Naresh, Harper, Keiser and Krumpe (2014) indicates that there are difficulties related to topics such as randomness, sample space and conditional and independent probability. Mathematics curricula denote a set of ideas that learners are taught and expected to learn. Therefore, teachers need to develop a strong, coherent, and intuitive pedagogical knowledge as well as simulation tools that will enable them to teach successfully and help learners understand probability concepts.

Game-based learning enriches the learning environment, improves the learners' performance, increases the learners' motivation, provides the opportunity to work within a group, and provides a fun learning environment (Hamalainen, 2008). Similarly, upon the increase in the importance of probability in teaching programmes, the teachers may be encouraged to expose the learners to the repeated trials of the same event, using concrete materials to understand the theoretical bases of probability (Batanero, Henry, & Parzysz, 2005). Through experimental research (such as repeated trials) learners may get the opportunity to be motivated and improve their stochastic intuitions as they are actively involved and eventually establish a sound understanding of probability, (Borovcnik & Kapadia, 2009a).

2.2.2.2 The four main reasons for using games in teaching and learning of mathematics

Ekonesi and Ekwueme (2011) outlined the four reasons for using games in teaching-learning mathematics: practice, motivation, anxiety and understanding. These reasons are discussed in detail as follows:

Firstly, is **practice** where games are used to teach probability to ensure that learners are practicing facts and formulae, even if their practice is limited to the classroom. This ensures that learners put the formulae into practice in common applications instead of reciting formulae through games only. Therefore, such practice helps learners to develop the ability to solve problems and build on their knowledge for higher-level processes.

Secondly is **motivation,** where many learners find probability tedious and boring. Therefore, using games to teach probability generates excitement, making probability a fun subject for learners. Interesting subjects make learners more willing to study. Similarly, competitive games motivate learners to study because they want to do well in the competitions.

Thirdly is the **anxiety,** as since probability creates anxiety for various age levels, games focus on fun rather than performance. Therefore, games are an excellent method of reducing math-related anxiety in learners. Games allow learners to realize that they are at various levels of competence in mathematics which helps to ease the anxiety that comes with the fear of being behind other learners.

Lastly is **understanding**, because learners may not fully understand certain probability concepts, or they may be able to understand concepts without being able to apply them. Using games to teach probabilities may help learners develop a better understanding of both concepts and applications.

Similarly, Davies (1995) summarised the advantages of using games in mathematical classrooms:

- The first advantage is **meaningful situations** where the application of mathematical skills is created by games.
- The second one **is motivation** where learners freely choose to participate and enjoy playing.
- The third advantage is the **positive attitudes,** as games provide opportunities for building self-concepts and developing positive attitudes towards mathematics by reducing the fear of failure and error.
- The fourth one is **increased learning** in comparison to more formal activities, as greater learning can occur through games due to the increased interaction between learners, opportunities to test intuitive ideas, and problem-solving strategies.
- The fifth advantage is about **different levels**, as games can allow learners to operate at different levels of thinking and to learn from each other. In a group of learners playing a game, one learner might be encountering a concept for the first time, another may be developing his/her understanding of the concept and a third may be consolidating previously learned concepts.
- The sixth one is **assessment** where learners' thinking often becomes apparent through the actions and decisions they make during a game, so the teacher has the opportunity to carry out diagnosis and assessment of learning in a non-threatening situation.
- The seventh advantage is **home and school** where games provide "hands-on" interactive tasks for both school and home.
- Lastly, **independence** where learners can work independently of the teacher. The rules of the game and the learners' motivation usually keep them on task.

Moreover, one can say mathematical games include mathematical values that help to develop mathematical thinking and can be integrated and used as a teaching strategy in the process of mathematical education.
This section describes probability in three perspectives: classical probability, which is deemed as a priori probability because of its alignment with the rationalist perspective; subjective probability as a degree of belief based upon personal judgement and frequentist interpretation which is the limit of relative frequencies in long trials. Probability terminologies were also identified as random phenomena, chance, likelihood, fundamental chances and equal chances. Lastly, probability games were highlighted as a significant way to teach probability because they encourage practice, motivation, ease anxiety, and promote understanding in learners. Consequently, there is a need for learners to stop looking for a formula to apply or a procedure to follow when approaching probability problems or tasks. Learners should, therefore, develop the mentality of thinking mathematically about the problem. The next section will look at the definitions of visualisation and visual probability teaching strategies.

2.3 VISUALISATION

2.3.1 Definitions of visualisation

Visualisation is a representation of an object, situation or a set of information as a chart or other image, or it is the formation of a mental image of something. Similarly, visualisation is the process of representing data graphically and interacting with these representations to gain insight into the data. Any technique for creating images, diagrams, or animations to communicate a message through visual imagery is an effective way to communicate both abstract and concrete ideas. Thus, the term visualisation is used to refer to both internal mental representations and external displays. Arcavi (2003) defines visualisation as :
The ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools, with the purpose of depicting and communicating information, thinking about and developing unknown ideas and advancing understandings. (p. 217)

This definition is complemented by Vavra et al.'s (2011, p. 22) explanation of the concept that states: “the term visualisation can be used to name a representation, to refer to the process of creating a graphical representation or as a synonym for visual imagery.” Visualisation as a graphical representation refers to:
pictures, three-dimensional models, schematic diagrams, geometrical illustrations, computer-generated displays, simulations, animations, videos, and so on. Objects can be displayed in

a variety of media formats, including paper, slides, computer screens, interactive whiteboards, or videos, and may be accompanied by sound and other sensory data (ibid.).

Similarly, the study by Presmeg (1995) articulates that visualisation includes processes of constructing and transforming both visual and mental imagery (p. 2). In support of the above definitions, the New Oxford American Dictionary defines visualisation as the process of forming a mental image or making (something) visible to the eye (Siirtola, 2007). Moreover, Makina (2010) affirms that visualisation is very important in ‘teaching for understanding’ in mathematics because it helps teachers to engage learners in realistic situations and with facilitation of lessons. Looking at all these various definitions of visualisation according to the above-mentioned authors, they all argue that visualisation is crucial in teaching for understanding and promotes mathematical thinking in learners.

Moreover, despite the existence of various definitions of visualisation, a very important aspect to consider is what visualisation entails from a pedagogical perspective. In this context, the process of visualisation in terms of teaching refers to the integration of visuals from the initial lesson planning stages to the lesson delivery or presentation. This includes the preparation of classroom activities/tasks which are meant to assess learners’ understanding of the concepts being taught, through the use of visuals. The lesson objective, which is also a major component of lesson planning, determines the type of visuals required at each stage of the lesson. The teacher is expected to develop appropriate visual materials or use ready-made materials, depicting key concepts of any given topic. The definitions of visualisation also imply two types of visualisation which are internal and external. The above-mentioned types of visualisation are discussed as follows.

2.3.1.1 Internal visualisation

Visualisation processes involve the use of imagery. Kosslyn, Thompson, and Ganis (2006) describes working with visual imagery as the ability to form mental representations of the appearances of objects and to manipulate these representations in the mind. Therefore, internal visualisation is a representation in the mind of an individual, derived from imagery or imagination, (Hegarty, 2004). It can be interpreted as the ability to mentally store and manipulate visual representations in the mind. For example, if one has ever dissected a human kidney and envisions what the kidney looks like, then one is relying on an internal or mental representation of a kidney. It is the mental image of the kidney accessed from memory. Presmeg (2006) argues that “the position is taken that when a person creates a spatial arrangement, there is a visual image in the person’s mind, guiding this creation” (p. 206).

Therefore, some type of mental imagery must initially occur for an external representation to occur. Similarly, De Koning and Van der Schoot (2013) stress that internal and external visualisation processes do not operate in isolation but they are interconnected. That is, both internal and external visualisation may develop a learners' understanding of probability concepts.

2.3.1.2 External visualisation

External visualisation is a representation in the environment that can be perceived by an individual, (Hegarty, 2004). It is a visual display that occurs in the world and comprises both static images such as drawings, graphs, charts and diagrams, and dynamic representations such as animations. For example, an image of a human kidney printed in a textbook is an external representation of a kidney. Therefore, diagrams as part of external visualisation further enable learners to show how information is related and organised, and they help work out a solution to the problem without errors of omission (Eysink et al., 2009). For Konyalioglu, Aksu, and Senel (2012) visualisation is a bridge built between the world of experiments and the world of thinking and reasoning. They further articulate that visualisation presented in the form of drawings, shapes and concrete models ensures a stronger conceptual perception as drawings about abstract concepts lead to mental interpretation. This study focused on the ability of teachers to construct mathematical models to facilitate teaching of probability concepts to promote mathematical thinking.

Having looked at the different types of visualisation discussed earlier in this section, it is clear that both forms play important roles in scientific and mathematical problem-solving (Ferguson, 1977; Miller, 1986). Therefore, the role of any type of representation is to communicate a mathematical idea in a way that is easily comprehensible by the learners and which conveys different meanings of a single concept. Both internal and external representations can be used by learners to solve problems (Matheson & Hutchinson, 2014).

As stated by Boaler, Chen, Williams, and Cordero (2016, p. 5) "teachers who emphasize visual mathematics and who use well-chosen manipulatives encourage higher achievement for students, not only in elementary school…" Visualisation in mathematics pedagogy refers to the deliberate use of visuals to promote deep understanding of concepts "both from teachers introducing mathematical ideas visually, and students using visuals to think and make sense of mathematics and connecting previously unconnected theories in mathematics" (Boaler, Williams, & Cordero, 2016, pp. 5, 7). Despite the vital role that visualisation holds in mathematics pedagogy, it is important to consider the role of both visuals and text in mathematics instruction.

Both text and visuals are important in teaching, but the general notion that the visual mode is less important and less formal has led to its devaluation in mathematics classrooms (Presmeg, 2006). The concept of visualisation is often associated with text information because the two are interrelated. Boerma, Mol and Jolles (2016) affirm that pictures can facilitate the creation of mental representation as well, as they can clarify implicit or unclear relations in the text. Stokes (2002) concurs that the connection of visual and verbal information is evident throughout history. This study argues that unfortunately, it is the disregard for visualisation in teaching that has contributed to text-dominated teaching strategies which sometimes cause learners to miss out on important aspects of a given concept, hence, the need to acknowledge the importance of visualisation in mathematics pedagogy.

Before one can decide to opt for visualisation and incorporate it into one's teaching, it is important to consider the importance of visualisation in teaching mathematics. Visuals are iconic and they often resemble the thing they represent (Avgerinou & Petterson, 2011, p. 7). A holistic and interdisciplinary approach is therefore required for the effective teaching of probability, especially at the secondary phase. Avgerinou and Petterson (2011) further assert that images speak directly to us in the same way experience speaks to us, that is, emotionally and holistically. This study adopted Arcavi's definition because it is very comprehensive and encompasses the overall objective of this study.

2.3.2 Visual tools for teaching probability

2.3.2.1 Manipulatives

Manipulatives are visual aids that assist concrete visualisation which helps learners to cope with stress linked to the learning of mathematics, (Ruzic & O'Connell, 2001). Using manipulatives helps learners to draw their learning from concrete materials, which help their limited mental maturity to grasp abstract mathematical concepts (Piaget, 1952). Using various manipulatives provides an exciting classroom environment positive, promotes learner attitudes toward mathematics learning, and greatly reduces anxiety (Ruzic & O'Connell, 2001). Apart from enhancing mathematical learning, learners are also given a chance to reflect on their experiences. Hence, manipulatives such as concrete manipulatives like coins, dice, marbles, decks of cards, spinners, coloured balls; and external representatives like Venn diagrams, Tree diagrams, contingency tables to mention a few; can be successfully used in introducing mathematics lessons, to practice, or to remediate mathematical concepts in mathematical instruction. The said concrete manipulatives may also be used in a game format to teach probability in a visual manner. This will only be possible if the manipulatives are appropriate for the learners and have been chosen to meet specific goals to increase learners' mathematical thinking and understanding, instead of learners simply moving the manipulative objects around.

Moreover, manipulatives ease learners' anxiety about learning mathematics and they are likely to evoke and heighten learners' interest in the learning of mathematics in a positive manner by creating a learning environment that is fun and enjoyable (Hunt, Nipper, & Nash, 2011). In turn, such an environment may help learners to relax and overcome any stress that might have arisen from learning mathematics as a subject. Golafshani (2013) reinforces the idea of using manipulatives to clarify difficult concepts by stating that the uses of manipulatives to clarify difficult concepts have some direct effects on struggling learners. This means that if there are concepts learners struggle to understand and comprehend, manipulatives can be effective artefacts to use in scaffolding the learning for them to understand and comprehend the concepts.

Although manipulatives can play an important role in the teaching of mathematics, they nevertheless need to be used carefully to create a strong understanding and justification for each step of a procedure (Clements & Sarama, 2016). Therefore, manipulatives support teaching and can provide mediation of learning. Manipulatives on their own do not carry mathematical ideas but the way they can be manipulated by learners can bring about meaningful understanding and learning.

It is therefore important to keep in mind that the manipulative objects and tasks given to learners are appropriate for the grade level being taught (D'angelo & Iliev, 2012). Teachers need to understand that even though manipulatives are physical, understanding how they present concepts nevertheless requires cognition. In short, a manipulative is just a physical representation of a concept, not the concept itself.

2.3.2.2. Challenges associated with manipulatives

According to Golafshani (2013) challenges that are associated with the use of manipulatives were identified, including teachers' beliefs when using manipulatives, i.e. some teachers feel that manipulatives took much time to introduce concepts during the lessons. Other teachers feel manipulatives waste time in teaching and therefore they are less important than the "serious" work of learning mathematics. Ultimately, manipulatives in these situations are not likely to be used, nor are learners being encouraged to make use of them. Another challenge that is associated with manipulatives has to do with the failure of teachers to match the manipulatives with the right level of instructional guidance as well as the age or grade of the learners' or their levels of understanding. Concurrently, arranging the classroom as a learning environment that allows manipulatives to be effectively used, seems to be another concern. When teachers fail to structure the learning environment (classroom) in ways that support learning with concrete materials, learners may equally fail to find the underlying concepts or processes, (Brown, Mcneil, & Glenberg, 2009). Therefore, the classroom set-up should be manipulative user-friendly.

In this study, the various visual tools mentioned above were used to construct mathematical models (such as coins to construct Tree diagrams, dice and coloured balls to construct contingency tables) that the teachers used to facilitate teaching in probability lessons. Interestingly, some teachers used various scenarios and used the information to construct Venn and Tree diagrams. Similarly, dice were also used in a game format to explore the notion of random events. Games can facilitate mastery of basic probability skills in a much more rapid way and with longer-lasting effects if that practice is done in the pursuit of a meaningful goal, rather than mere repetition for its own sake (Devlin, 2011).

2.3.3. The four steps/procedures for teaching probability

Probability is one of the mathematics topics that is regarded as difficult by most of the high school learners, (Dollard, 2011). Therefore, when teaching probability, sufficient education and support may be needed to develop intuitions for both teachers and learners (Batanero, Contreras, Fernandes, & Ojeda, 2010), which may increase learners' levels of interpreting what they see.

The studies carried out on probability teaching have recommended using computers as a way to understand abstract or difficult concepts and to increase learners' talents (Gaise, 2005; Gurbuz, 2007; Koparan, 2016; Koparan & Yilmaz, 2015; Mills, 2002). Similarly, Batanero, Henry, and Parzysz (2005) emphasize that learners should execute the simulations to help them solve simple probability problems, which are impossible through physical experiments in computer courses at school. Simulation is described as the most suitable strategy in focusing better on concepts and in decreasing technical computations (Borovcnik & Kapadia, 2009b). Therefore, simulation provides an opportunity to strengthen understanding statistical ideas (Konold, Harradine, & Kazak, 2007) and to support the learning process while studying an experiment of chance.

In this study, I will focus on the four probability teaching strategies as discussed below. Batanero, Chernoff, Engel, Lee, and Sánchez (2016b) suggest four probability teaching strategies that they argue can be used to promote probabilistic reasoning. Probability reasoning is a mode of reasoning that refers to judgements and decision-making under uncertainty and is relevant to real life, for example, when evaluating risks (Falk & Konold, 1992). It can also be defined as thinking in scenarios that allow for the exploration and evaluation of different possible outcomes in situations of uncertainty. In this section, I look at the following teaching strategies:

- First is **identifying random events in nature, technology, and society (IRE):** This may include random process or experiments that show various interpretations and misconceptions held by learners. The emphasis is on the need to reinforce understanding of randomness in learners which leads to teaching from the known to unknown.
- Second is **analysing conditions of random events and derive appropriate modelling assumptions (ACR):** Learners are developing awareness of themselves as legitimate creators of mathematical knowledge. The teacher should be able to facilitate the probability lesson, by allowing learners to be actively involved in the discussion and solve challenging probability problems. Probability models constitute important tools to recognize and solve problems and enable learners to critically asses the application of probability models of real phenomena. Learners work in groups to set up experiments and the teacher uses different levels of scaffolding and constraints to teach a problem.
- The third is **constructing mathematical models for stochastic situations and explore various scenarios and outcomes from these models (CMM):** Creating a mathematical model and working with it may help learners to find ways to partition the probability events. The teacher leaves the problem quite open-ended to carefully observe learner activity, then to pull the class back together frequently to capitalize on critical moments related to probability. This approach is directed at developing mathematical thinking, shaping ideas and making connections between probability concepts coherently and sensibly. Relevant and meaningful teacher talks during probability lessons involve drawing out the specific mathematics ideas encased within students' methods, and sharing other methods and advanced students' mathematical thinking of appropriate mathematical conventions.

- Lastly is **applying mathematical methods and procedures of probability (AMM)**: The teacher should be able to administer practical investigations and projects to learners and let the learners present their work in class. Teachers encourage learners to keep working or thinking about a problem, give them instrumental help that facilitates their progress, allow plenty of time for students to complete work and require learners to go back and try again when they have reached inadequate solutions or encourage them to cope with multiple strategies. In the next section, the study deal with the term 'mathematical thinking' as a less diluted mathematical perspective and how it can be enhanced in the teaching of probability.

This section discussed on how probability is viewed as difficult topic by most high school learners, (Dollard, 2011). Therefore, this study focuses on using visual probability teaching strategies that may enable learners to understand the probability concepts as they visualise through the teachers' constructed mathematical models. An important step in an application of probability to real world phenomena is modelling random situations (Chaput, Girard, & Henry, 2011).

2.4 MATHEMATICAL THINKING

2.4.1 Definitions of mathematical thinking

Learning to think mathematically is not about getting answers (Devlin, 2012), though once you have learned to think mathematically, getting the right answer becomes a lot easier than when you are just following procedural recipes. Mathematical thinking may be viewed as a process of trying and reflecting but not getting answers. In relation to this Devlin (2011) states that "mathematical thinking is a whole way of looking at things, of stripping them down to their numerical, structural, or logical essentials and of analysing the underlying patterns" (p. 59). It is about knowing how and in what way to think that provides mastery in most areas of an individual's life. Also, individuals at every stage of their lives use mathematical thinking, consciously or unconsciously, to solve problems they encounter. Therefore, mathematics is one of the most significant tools that is known to improve thinking (Drijvers, Kodde-Buitenhuis & Doorman, 2019).

Drijvers, et al. (2019) further added that mathematical thinking enables learners to establish a connection between theory and practice, that is, correlate theory with things they see around them regularly and acquire the habit of doing mental math, which will be essential in everything that they will do in the future. Lastly, once mathematical thinking is fully developed, it may enable learners to solve mathematical problems in topics such as probability in a smarter way by breaking them into smaller parts and thinking in a creative way to arrive at a smart solution in the shortest possible time (Drijvers, et al., 2019).

However, the ability to think mathematically is not the same as 'doing mathematics'. Surprisingly, mathematical thinking includes logical and analytical thinking as well as quantitative reasoning and all crucial abilities (Devlin, 2012). At times, mathematics goes from being confusing, frustrating and seemingly impossible, to make sense and of being hard, but doable. Therefore, mathematical thinking skills are crucial mental abilities required to succeed in many professions and walks of life, and which require learners to master mathematical concepts such as probability. Wood (2001) states that:

> Learning mathematics with understanding is thought to occur best in situations in which children are expected to solve problems, reason, and communicate their ideas and thinking to others. Moreover, it is thought that situations of confusion and clash of ideas in which learners are allowed to struggle to resolution are precisely the settings that promote learning with understanding (p.116).

Wood sees the heart of reform as a transformation in the ways teachers teach mathematics and that the ways of learning and teaching result in learners knowing a different kind of school mathematics. One of its by-products is a mathematics learner who can reason.

A learner who is good at reasoning can adequately explain his/her thinking and do more than just list the procedures or summarise the answer. Such a learner can use data to make, test, or argue a conjecture. He/she can speculate, test ideas and defend or argue them through contextualised problem-solving tasks (Diezmann, Watters, & English, 2001). Thus, problem-solving, mathematical communication and good mathematical reasoning are probably equally important characteristics of a successful mathematical thinker. The ability to explain what one is thinking mathematically and clarify one's thinking and the thinking of others increases one's understanding of a concept.

Conversely, being able to get a clear idea of what a learner is thinking is often difficult unless a good explanation and representation of the solution are provided. Clarke, Goos and Morony (2007) state that developing an appropriate visual representation of the information in a problem is crucial to successful problem-solving. This may be another identifying characteristic of successful mathematical thinking. Learners need to practice, however, in presenting and defending their answers and should be given repeated chances to show what they are thinking and how the problem was solved, if they are to improve at such skills.

Moreover, in support of what Clarke et al. (2007) have discussed, various forms of visualisation can be used for explaining probability concepts such as rules, probability problems, images, sketches, schemas for ordered listings of random experiment results, (contingency) tables, Venn diagrams, Tree diagrams, circular diagrams, tangrams (unit squares), coordinate systems and stochastic node graphs. These various forms of visualisation have different structural aspects or properties that determine the range of their usability. For example, Tree diagrams are naturally suited for solving sequential problems like multiple coin tosses or throwing dice or for listing all possibilities, while Venn diagrams are suitable for representing compound events, (Lukáč & Gavala, 2019).

These representation strategies have an unofficial list of ways to present the problem and its solution that expresses thinking in a variety of ways – for example words, drawings, pictures, charts or graphs and written explanations. Costa and Kallick (2000) state that these kinds of thinkers use representations to help show exactly what he or she was thinking when figuring out a problem and arriving at a solution. Costa and Kallick (2000) further add that successful mathematical thinkers notice how ideas are related: making higher-level connections allow the learners to draw forth a mathematical event and apply it to a new context in a way that connects familiar ideas with new concepts or skills (Costa & Kallick, 2000).

From my point of view, it is hard for a senior teacher to do something so new that makes him/her feel like a novice teacher. Senior teachers in Namibia are not trained on how to teach mathematical thinking. Therefore, I quote from an old saying as follows:

Thus a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his learners in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his learners by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent mathematical thinking. (Polya, 1945, p. V).

2.4.2. How learners' development of mathematical thinking can be facilitated

There are many varied windows through which mathematical thinking can be viewed. Being able to use mathematical thinking in solving problems is one of the most fundamental goals of teaching mathematics (Stacey, 2006). In this study, I further looked at Carney et al., (2014)'s observation instrument that was developed for developing mathematical thinking . This instrument is designed to measure five dimensions related to engaging learners in meaningful, intellectual activities that provide an authentic understanding of mathematics and build connections both within and outside the classroom. The following are specific dimensions within the Developing Mathematical Thinking (DMT) framework:

Taking learners' ideas seriously (TLI) involves valuing and building upon learners' intuitive understanding of mathematical concepts (Carpenter & Lehler, 1999). For example, when learners solve an unfamiliar yet meaningful probability problem, they draw on their prior knowledge and experience. Therefore, teachers can connect learners thinking to more efficient and abstract methods (Van Galen & Gravemeijer, 2003).

Encouraging multiple strategies and models (EMMS) involves developing learners' understanding of various models, representations, and approaches to solving problems. For example, when learners generate, evaluate, and utilize different probability strategies and models, they recognize there are many ways to solve problems and represent solutions, (Carney et al., 2014).

Pressing Learners Conceptually (PLC) focuses on building connections between mathematical strategies and models and progressively formalizes those ideas and methods for solving problems, (Forman, 2003). This occurs, for example, once students have had the opportunity to work on their solution methods and generalize to new situations they relate these to formal probability terms and conventions.

Addressing misconceptions (AM) are valuable tools to build mathematical thinking. Making mistakes and learning from them is an integral part of doing mathematics at any level. These mistakes often recur even after teachers have demonstrated a correct procedure because they stem from deeper probability misconceptions. By being aware of why and how probability misconceptions occur, teachers can move students – through models and discussions – to a deeper level of probability conceptual understanding that precludes such mistakes. Additionally, mistakes can be opportunities for learners to engage in justification, evaluation, and inquiry (Borasi, 1987).

Focusing on the structure of the mathematics (FSM) involves facilitating learners' understanding of fundamental, or structural, probability concepts. Focusing on these structures allows learners to build an understanding of, and establish connections between, these fundamental concepts and the particular topics being studied. Understanding probability structural components can help teachers tie different concepts together both within and across grade levels, rather than teaching topics in isolation. Moreover, when instruction does not focus on the structure of probability concepts, learners often rely on memorized tricks or formulas and have difficulty solving complex problems or applying mathematics to new situations, (Carney et al., 2014).

Visual tools help learners to develop new ideas in probability concepts. Teaching probability provides learners with understanding and develops their critical and mathematical thinking about the role of probability in their lives and in mathematics. Therefore, mathematical thinking enables learners to identify areas from real life where mathematics is applicable. Learners who think mathematically are also able to think outside-the-box and solve probability problems using visual tools.

This section discussed the teaching of probability and how the manipulatives were employed for visualisation processes in probability lessons. These manipulatives include any concrete materials like spinners, dice, marbles, coins, cubes and decks of cards to mention a few, that help to facilitate the teaching of probability processes. Furthermore, manipulatives can also be used to play educational games with the aim of facilitating the teaching in probability classrooms. However, there are also challenges associated with the use of manipulatives, such as teachers not having time to prepare visual tools, that results in them not using them or encouraging learners to use them. Similarly, another challenge discussed was the failure of teachers to match the manipulatives with the right level of instructional guidance, the age or grade of their learners or their level of understanding.

The section further touched on the internal and external forms of visualisation processes that are interrelated and crucial for teaching probability concepts. Hence, four probability teaching strategies were suggested with the focus on promoting probabilistic reasoning, involving judgements and decision-making under uncertainty and which is relevant to real-life and which may encourage mathematical thinking in learners. This study shows that mathematical thinking helps to establish a connection between theory and practice. Therefore, mathematical thinking indicators were discussed to alert teachers on how to promote mathematical thinking within probability lessons to ensure understanding in learners.

From what has been discussed above, one can conclude that learning to think mathematically is not only about getting answers, but it is about trial and error and a process of trying and reflecting, yet thinking outside the box. In the next section, I look at the theoretical framework of the study.

2.5 THEORETICAL FRAMEWORK

2.5.1 The Dual Coding Theory

This study is positioned within the Dual Coding Theory (DCT) as proposed by Paivio and attempts to give equal weight to verbal and non-verbal processing. Paivio (1991) states that: Human cognition is unique in that it has become specialised for dealing simultaneously with language and with non-verbal objects and events. Moreover, the language system is peculiar in that it deals directly with linguistic input and output (in the form of speech or writing) while at the same time serving a symbolic function for nonverbal objects, events, and behaviour. Any representational theory must accommodate this dual functionality (p. 53).

Dual Coding Theory as proposed by Paivio (1986) advocates the use of both verbal (mathematical symbols) and non-verbal (visuals) cues in teaching to enhance learners' mathematical thinking. The theory identifies three types of processing.

Representational processing is the direct activation of verbal or non-verbal representations. Mental representations are associated with theoretically distinct verbal and nonverbal symbolic modes and retain properties of the concrete sensorimotor events on which they are based. The **verbal system** contains visual, auditory, articulatory, and other modality-specific verbal codes. Moreover, **verbal codes** retain their separate and discrete identities even when connected in hierarchies or other associative networks; thus, they are processed serially or sequentially. Conversely, **nonverbal representations** include modality-specific images for shapes, environmental sounds, actions, skeletal or visceral sensations related to emotion, and other non-linguistic objects and events. Such imaginal representations are analogous or perceptually similar to the events that they denote, rather than being arbitrary symbols. Similarly, mental images evoked by emotionally laden words or phrases have visceral properties similar to those experienced when one is actually in the presence of an effective object. **Nonverbal representations** can encode information in parallel or simultaneously.

Referential processing is the activation of the link between the verbal system and a non-verbal system or vice-versa. The joined corresponding verbal and imaginal codes potentially allow such operations as imaging to words and naming to pictures. Referential connections in the opposite direction, from images to names, permit learners to label pieces of science apparatus or biological objects and their parts and other figural information.

Associative processing is the activation of representation within the same verbal or non-verbal system. Thus, associative connections join representations with verbal and nonverbal systems. Therefore, within a verbal system, words are joined to other related words. Additional examples of verbal associations include: connections between instance and category names;

teachers and learners encoding a lesson as an associative chain of key terms and phrases; or the schematic representation for a lab report as a linked series of labelled parts. Within the nonverbal system, associative connections join images to other images in either the same or different sensory modalities. Similarly, a visual image of a Bunsen burner can be associated with visual images for other objects in a science experiment. Learners also link successive images for concrete events reported in narratives and other concrete school materials (see Figure 1). As a result, individual verbal and image representations vary in their activity levels, with some representations highly active and others decreased at any given time. Strong activity may be associated with conscious nonverbal and verbal experiences. Therefore, a given task may require any or all of the three kinds of processing. According to Mayer and Anderson (1991, p. 485):

> "This theory predicts that learners will remember and transfer material better if they encode the material both visually and verbally because they have two separate ways of finding the information in memory."

The over dependency of teachers on one mode of presenting subject content to learners can be detrimental to learning as some learners are visually inclined while others are verbally inclined.

The figure below shows the representational units with their referential (between- system) and associative (within-system) interconnections.

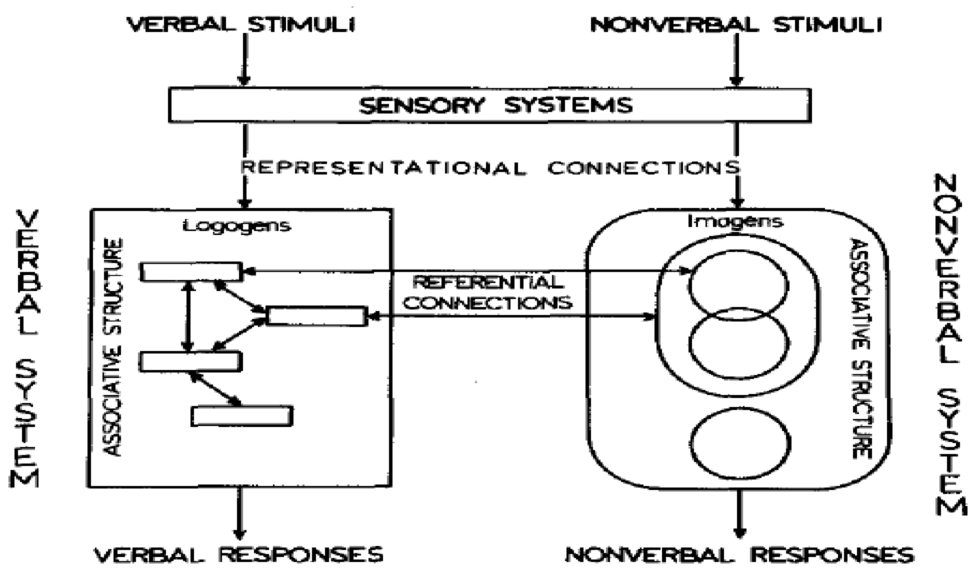

Figure 2.1: Verbal and non-verbal symbolic systems of Dual Coding Theory. A dual coding approach by Paivio (1986).

Clark and Campbell (1991) used dual coding mechanisms to develop a general theory of number processing. Dual Coding Theory emphasises the concrete basis of number concepts and the roles of associative mechanisms and imagery in performing numerical operations. Similarly, basic dual coding processes have been used to teach arithmetic, whereby learners are first taught to name numerals which are associated with a group of objects and pictures and then later their meanings.

Interestingly, Tuley and Bell (1997) developed a remedial mathematics programme for learning processes. Similar to the JUMP (Junior Undiscovered Math Prodigies) programme described below, this remedial programme emphasises concretisation of mathematical operations and concepts. However, it differs from other programmes because it teaches learners how to use visualisation (mental imagery) to represent numbers and operations. Furthermore, the programme proceeds from concrete experiences using number lines and cubes to image the entities and operations, to computation. These steps are known as the 'mathematics ladder', where learners have to climb, step by step, at their own pace. The evidence shown strongly supports the effectiveness of the programme (Tuley & Bell, 1997).

In support of DCT, mathematicians like Mighton (2003) started an educational charity called JUMP (Junior Undiscovered Math Prodigies) for elementary school learners who experience difficulty in learning mathematics.. The programme involves systematic concretisation of mathematical concepts and operations many of which use pie charts and box diagrams that are familiar to all teachers. Within this programme, operations are applied to one problem and used systematically so that learners may master them before they move on to a complex one. Evidence shows that individuals and whole classes classified as slow learners excelled in mathematics as a result of JUMP intervention (Mighton, 2003).

Having looked at the above explanations for DCT, it implies that the theory has been applied to many cognitive phenomena including mnemonics, problem-solving, concept teaching and language.

2.6 CONCLUSION

The reviewed literature indicates that the use of manipulative objects began in ancient times. Much literature supports the use of manipulatives in mathematics instruction. Valuable learning occurs when learners actively construct their mathematical understanding which is accomplished through the use of visual tools, (Boggan, Harper, & Whitmire, 2010). However, integrating hands-on games (visual tools) in mathematics lessons is not enough; teachers need to correctly plan and develop teaching and learning aids based on lesson objectives to teach successfully and make learners understand probability and other mathematical concepts which are difficult for them.

In the Namibian context, few pieces of research have been done on the use of manipulative objects in mathematics, in spite of probability being a challenging topic (Ministry of Education, 2014). Therefore, this study seeks to explore visual probability teaching strategies to enhance mathematical thinking in Grade 11 classrooms.

In general, this chapter looked at the definitions and types of probability as classical, frequentists, and subjective. It also looked at the definitions of visualisation (which includes internal and external visualisation) and visual teaching strategies, including games as well as manipulatives used in the teaching of probability (e.g. coins, spinners, and dice). The chapter further looked at mathematical thinking definitions, the indicators for mathematical thinking as well as how mathematical thinking can be enhanced. Lastly, the chapter discussed the theoretical framework.

CHAPTER 3
METHODOLOGY

3.1 INTRODUCTION

This chapter describes the research methodology that underpins this study's research process. I adopted the mixed-methods design within an interpretive paradigm that enabled me to gather and analyse quantitative and qualitative data, while utilising a case study as my research method. I present the instruments that I employed in collecting data to answer my research questions. To fulfil this task, my research used the data collection tools such as the questionnaires, classroom observations and interviews. It also discusses the selection criteria of participants and how I analysed the data. The chapter concludes with an articulation of validity issues and ethical aspects used to intensify the quality of this research study.

In this section, I describe the research goals and objectives of the study. I also outline the research questions in connection with the types of data generation techniques used in this study.

3.2 RESEARCH GOALS AND QUESTIONS

The overall goal of this study was to explore how visual probability teaching strategies can be used by the selected mathematics teachers that participated in the intervention programme to teach probability for enhancing mathematical thinking in Grade 11 learners.

This study was inspired by a suggestion in the literature that "Probability is the only reliable means we have to predict and plan for the future, it plays a huge role in our lives, so we cannot ignore it, and we must teach it to all future citizens" (Devlin, 2014,p.ix)

3.3 RESEARCH ORIENTATION

3.3.1 An interpretivism paradigm

This research study was done within the interpretive paradigm. "The way in which one sees the world influences, the way in which one researches the world" (Bertram & Christiansen, 2014). This interpretive paradigm emphasizes that "the view of the world we see around us is the creation of the mind" and "we can only experience it personally through our perceptions which are influenced by our preconceptions, beliefs, and values" (Walliman, 2011, pp. 21 -22). This study intended to gather "insight and understandings of behaviour …. [and] explain actions from the participant's perspective" (Scotland, 2012, p.12) when they teach probability.

With this study, current practices in mathematics classrooms are better understood through the participants' interpretations and justifications of their actions in terms of their epistemological positions and pedagogical perspectives. Since this study is concerned with exploring visual probability teaching strategies for enhancing mathematical thinking, it is important to understand the participants' positions about these processes. Hence the appropriateness of the interpretive paradigm as the overarching philosophical lens for this study.

Creswell (2003) claimed that interpretivist researchers discover reality through their participants' views, their backgrounds and experiences. On the same note, a research paradigm is a representation of a particular world view that defines for the researcher who holds this view, what is acceptable to research and how this should be done (Bertram & Christiansen, 2014). Furthermore, Williamson (2006) refers to the interpretive paradigm as the knowledge that is constructed from observations that are made in a real and natural setting, making this paradigm well-suited for this study.

Interpretivism can be defined as a way of understanding human meanings and behaviour, without intervening in the process (Cohen, Manion, & Morrison, 2011a). Besides this, the interpretive research paradigm views reality and meaning-making as socially constructed; it holds that people make sense of social realities (ibid., 2018). In this study, I thus worked within the interpretive paradigm in observing how the teachers were using visual tools to teach probability in the interest of promoting mathematical thinking.

3.3.2 A mixed-method approach

A mixed-method is a research approach in which qualitative and quantitative data or techniques are mixed in a single research study (Turner & Ireson, 2010). Johnson, Onwuegbuzie and Turner (2007) view mixed-method research as a type of research design in which qualitative and quantitative approaches are used in identifying types of questions, research methods, data collection and analysis procedures. In this study, a mixed-method approach was adopted. This method further allows for contextual interpretations, the use of multiple methods and flexibility in choosing the best strategies to address the research questions.

This study employed both quantitative and qualitative methods of data collection; however, the qualitative approach was dominant. Quantitative data was obtained from a survey questionnaire with the intention to follow up the results from the survey and explore more in depth, so the qualitative sample included participants from the quantitative sample. The survey data was analysed and the results were used to plan for the qualitative data. Furthermore, qualitative data was obtained from video recordings of classroom observations of the teachers, reflective journals and reflective interviews with participants that were conveniently and purposively selected from three secondary schools in the region.

Drawing from Harris and Brown (2010) a mixed-method approach enables the researcher to answer confirmatory and explanatory questions. Johnson et al. (2007) further highlight that mixed-method research has advantages over other forms of research. Firstly, the combinations are used for confirmation of each other to provide a much richer context to the data. Secondly, the combinations are used to initiate new modes of thinking by attending to the paradoxes that emerge from the data sources. For Christensen, Johnson and Turner (2015), a mixed-method approach allows multiple sources and methods of data collection, and this ensures further data validation.

In this study, I used a mixed-method approach because it enabled me to get a more comprehensive and complete understanding of phenomena that could not be obtained with a single-method approach. The mixed-method approach allowed me to answer complex research questions more meaningfully, combining particularity with generality (Cohen et al., 2011b).

3.4 RESEARCH METHODS

This study basically made use of two methods. The first method was a survey of fifty Grade 10-12 secondary school mathematics teachers in the Khomas region, while the second method was a case study of three selected Grade 11 teachers that participated in an intervention programme.

3.4.1 Survey

A survey was the first method of this study. According to Cohen et al. (2011) surveys are a means of gathering data at a particular point in time with the intention of describing the nature of existing conditions or identifying standards against which existing conditions can be compared. On the same note, Creswell (2014) also writes that with a survey, there is rapid turnaround in data collection.

The survey questionnaire was divided into parts A and B respectively (see Appendix G). Part A was a general section with questions that sought the teachers' profiles while part B consisted of questions that were designed to ascertain how teachers in the region taught probability before they were introduced to the intervention programme. The survey helped me to find out the types of visual tools the teachers used, how and what extent they use them. This data helped me to craft the intervention programme.

3.4.2 Case study

Moreover, this study followed a case study developed from the secondary mathematics teachers' perspectives of practical activities and the use of visual tools within the Grade 11 classrooms in the Khomas region.

According to Creswell (2014), a case study is an in-depth exploration of a bounded system (for example an activity, event, process, or individuals) based on extensive data collection. Creswell (2008) posits that case studies can involve multiple cases where more than one case gives insight into the phenomenon being studied. Furthermore, case studies allow an in-depth understanding of the case (Creswell, 2008). In this study, I made use of a case study to explore how the selected teachers used visual tools for enhancing mathematical thinking in Grade 11 classrooms to find out their views and experiences of using these visual tools to teach probability. A case study was appropriate to my study because I researched a specific case – the three participants' visual probability teaching strategies for enhancing mathematical thinking. To find out about these strategies, I looked into the teachers' views and experiences of probability. Cohen, Manion, and Morrison (2017) highlight that case studies can establish cause and effect. Indeed, one of the strengths of case studies is that researchers observe effects in real contexts, recognising that context is a powerful determinant of both causes and effects.

3.5 POPULATION AND SAMPLING

Sampling involves making decisions about which people settings, events, or behaviours to include in the study (Bertram, & Christiansen, 2015). Purposeful sampling was used to select participants in this study. According to Creswell (2014), purposive sampling means that the researchers make specific choices about which people, groups, or objects to include in the sample. The word 'purposive' indicates that the sample is chosen for a particular purpose (Bertram, & Christiansen, 2015).

Mertens (2005) asserts that researchers typically select samples with the goal of identifying information-rich cases that will allow them to study a case in-depth when working within the interpretive paradigm. In this study, three participants that are chosen have atleast five (5) years of teaching experience and therefore deemed suitable for the study as they have been in the teaching profession and they are exposed to different classroom situations. These participants were also willing to take part in the study. The following sections describe in detail who participated in this study, where the research took place, how it was conducted and what steps I took in selecting participants of the study.

3.5.1 Grade 10-12 mathematics teachers as survey participants

Before the new curriculum was revised in 2019, Grades 10 and 12 (exit grades) wrote external examinations at the end of every year. Examiner reports for the past five years show underperformance in probability for both junior and secondary levels (see Table 1.1 in Chapter 1). Therefore, this necessitated the need to pilot the study with both the Grade 10 and 11 (new curriculum exit grades) mathematics teachers to find out how they use visuals to teach probability concepts.

Being a teacher in the Khomas region, I chose to conduct my research in the same region. Fifty questionnaires were distributed to all Grade 10-12 mathematics teachers in the region, but only thirty-four teachers responded and returned their questionnaires. The data from the survey questionnaires aided me to answer the research question 1.

3.5.2 The three grade 11 mathematics teachers as participants

After the survey, there was an intervention programme that involved two workshops with the Grade 11 mathematics teachers. During the induction workshop, I introduced the study to the mathematics teachers by acquainting them with the concept of visualisation in probability, what it entails, and how it can be used to promote mathematical thinking of probability concepts, Currently, the Namibian new curriculum for mathematics in the Grade 11 syllabus outlines several topics under the broad goal that requires learners to **understand and use probability.** The achievement of this goal is assessed through learners' ability to proficiently engage in probability calculations involving single and combined events.

The following topics were discussed during this workshop, focusing on understanding the depth of the following topics as per the Grade 11 syllabus and on enhancing mathematical thinking: probability of a single event; the probability scale from 0 to 1; addition and multiplication of probabilities; application of probability; simple and combined events; possibility diagrams; and Tree diagrams and Venn diagrams. Participants were asked to go think about what visual tools should be used for each topic and how to use the identified tools. The researcher played the role of the facilitator and participant. All participants and the researcher were actively involved in the discussions. This was about an hour's workshop and each participant came up with a reflective journal to reflect on what was discussed in this workshop.

In my study, three Grade 11 mathematics teachers gave consent to take part in the study. I purposively selected the three teachers from three different schools because they have all been teaching for more than five years and I believed they had had enough exposure to probability lessons. This put them in a more satisfactory position to demonstrate their teaching (because of their sufficient knowledge and ability to express their views and experiences through interviews). These three teachers were also conveniently selected because their schools were close to my school and that made my movements to their schools for observation easier. Two of these schools share a fence, while the third school is about 900 metres away. Furthermore, the three mathematics teachers chosen for this study showed interest and willingness to take part in the study during the intervention programme's first workshop. I chose to do my research with the Grade 11 mathematics teachers because several examiners' reports from 2011 to 2016 indicated that (see Table 1.1 in Chapter 1) most grade 10 & 12 (exit grades) learners have difficulties with understanding probability concepts. Having these Grade 11 mathematics teachers as participants to present and facilitate the lessons allowed me to observe various teaching strategies during the research process so that I could improve my practice.

We therefore planned the second workshop with the participant teachers about probability concepts to be taught and the visual tools to be used. These workshops were not video recorded. During this stage, I requested permission from the Director of Education as well as from the school principals where the intervention was conducted. The three schools were then identified and selected, from which three participants were chosen using the criteria described above.

3.5.3 Profiling and coding of participants

During the entire research process, teachers were given pseudonyms to keep their names anonymous for the sake of confidentiality. I also used the codes for their lesson observations to help me with analysing the data: Av_1 represented Mr Assery's video for lesson 1 while ASRI represented Mr Assery's stimulus-recall interview. All the participants were interviewed once. Table 3.1 Shows the teachers' profiles and coding.

Table 3.1: The teachers' profiles and coding

Teachers names	Qualifications	Area of specialization	Years of experience	Number of lessons taught	Gender	Classroom observation coding	Interview coding
Mr Assery	Bachelor of Education	Mathematics Chemistry	9	3	Male	Av_1 Av_2, Av_3	ASRI

Ms Emanu	Bachelor of Education	Mathematics	20	3	Female	Ev_1 Ev_2 Ev_3	ESRI
Ms Silvy	Honours in Education	Mathematics and Physical science	12	3	female	Sv_1 Sv_2 Sv_3	SSRI

The study did not focus on gender issues; the indicated teachers are those that met the criteria requirement.

3.6 RESEARCH DESIGN

A research design is defined as:

> "a plan or strategy that moves from the underlying philosophical assumptions to specify the selection of participants, the data gathering methods to be used and the data analysis to be done" (Maree, 2020, p. 80).

The data in this study was collected into six phases which consisted of the survey, the induction workshop, main workshop, pilot study, implementation (teaching), and interviews.

3.6.1 Phase 1: Survey questionnaire piloting and distribution

Survey data is used to describe and explain the status of phenomena, to trace change and draw comparisons (McMillan & Schumacher, 2001). In this study, I first designed a questionnaire and piloted it with the mathematics teachers at the school in which I am currently teaching. This enabled me to check if questions were clear and not ambiguous. Some teachers revealed that the phrase 'visual tools' was not familiar to them. I therefore made some adjustments by adding the definition for visual tools in the questionnaire. This made the questionnaire more suitable for the main study (See Appendix G). This was intended to make sure that the validity and effectiveness of the questionnaire to generate appropriate data was enhanced.

After piloting, I distributed fifty questionnaires to all Grades 10 to 12 mathematics teachers in the Khomas region through their circuit representatives. Initially I managed to get twenty questionnaires back before the schools went on lockdown due to the Coronavirus pandemic. The lockdown took about three consecutive months that resulted in teachers misplacing and losing the questionnaires. When the schools re-opened in June 2020, I was among the teachers that set a regional examination paper for Grade 11 that was to be written in the

August mock examination. I took advantage of this chance and distributed another twenty questionnaires, but because the session was so hectic, only ten teachers managed to complete the questionnaires during that time and submitted them to me. Other teachers promised to send them via email, of which I received four later in the same week. My interest in this phase was in the teaching methods and the types of visualisation tools that the teachers used to teach probability to promote mathematical thinking before the intervention programme. The survey served as one of my primary data collection tools. The results of the survey helped me to answer research question 1 and to craft the intervention programme.

3.6.2 Phase 2: The induction workshop

In this phase, I invited seven teachers from different schools in our circuit to an induction workshop where I introduced them to my study. The aim for this workshop was to create awareness of using visuals to teach probability for promoting mathematical thinking. We discussed various probability concepts with their objectives. Towards the end of this workshop, I explained the selection criteria to the teachers, which resulted in them volunteering to participate. We then planned workshop 2 which was about planning topics to be taught. This workshop was one-hour long**.**

3.6.3. Phase 3: Main Workshop (Planning)

In this phase, I conducted the workshop 2 with the three selected teachers and we planned the sub-topics to be taught, prepared lesson plans, and identified the materials. This involved a discussion on the use of visual tools (as indicated in the induction workshop) in the teaching of probability in a Grade 11 classroom for four months**.** Discussions and hands-on tasks were some of the activities tackled during the planning workshop. The topics selected were **probability of a single event, the probability scale**, **simple combined events** and **teaching probability in practice**. The lessons included teaching strategies that made explicit use of visual tools. I introduced the teachers to some of the approaches to teaching probability (for example, one can make use of coins or dice to construct a mathematical model like a contingency table). These models can then be used to facilitate learning. Teachers participated actively during this discussion.

My role as a researcher was that of a participant-observer. Participant observer enters the classroom and integrates with the learners directly. Burns and Kim (2000, p. 82) states that “The researcher becomes a member of the context and participates in its culture and activities.” After the discussion, we came up with a carefully designed common lesson plan showing clearly what was to be done in all three lessons for each participant (see Appendix

J). The participating teachers received the visual tools as per their request, like the coloured balls and cubes, decks of cards, dice and coloured pins that they used during the workshop and circulated during the implementation process. Teachers were allowed to provide the visuals of their choice too. In this workshop, each participant produced a reflective journal in which they gave an account of their work in progress and reflected on their experiences during workshops and later in their teaching.

3.6.4. Phase 4: Pilot study

In the Namibian revised curriculum, Grades 10 and 11 is a two-year course and therefore lessons were piloted using the grade 10 teachers and their classes at my school. Piloting of the lesson helped to check the use of video recording instruments and the analytical tools.

3.6.5 Phase 5: Implementation (teaching)

The lesson implementations took place between the second school term of 2020 and the first school term of 2021. Each teacher was observed teaching his/her lessons planned in Phase 2. The focus of this intervention was to teach probability using visual tools. In this phase, I was supposed to observe my participants at mutually agreed times as they presented their lessons using visual tools, but as a result of the COVID-19 pandemic (that affected various countries including Namibia), schools were put on lockdown for about three consecutive months. I could not observe all the teachers in the year 2020 due to the limited time, so I had to continue observing the last two teachers the following year (2021). My role at this stage was that of an observer. I observed and video recorded three lessons per teacher. My focus was on how teachers made use of the various visuals provided in workshop 2, to develop models and teach probability for promoting mathematical thinking**.**

I observed how the participants used visual tools like balls, cubes, dice, decks of cards and coins when teaching probability and took field notes as I observed the participants' lessons. All eight lessons were video recorded, while one of the lessons was audio recorded due to technical problems which arose. The aim was not to compare teachers, but to see how visual tools were used by the teachers in the lessons taught and share the knowledge and skills with other participants. This data helped to answer research question 2.

Table 3.2 Shows the implementation process for the intervention programme.

Table 3.2 Shows the implementation process for the intervention programme

Probability concepts taught	Objectives	No. of lessons to be covered per topic	Lesson duration
Probability of a single event and the probability scale (0-1)	Understand and use probability	One lesson per participant	40 minutes each lesson
Simple and combined events	Understand and use probability	One lesson per participant	40 minutes each lesson
Possibility diagrams; Tree diagrams and Venn diagrams	Understand and use probability	One lesson per participant	40 minutes each lesson

3.6.6 Phase 6: Interviews

Stimulated-recall interviews followed after the three lesson presentations for each participant were conducted. Each teacher was interviewed once to establish the participants' views and experiences on how and whether the use of visual models promoted mathematical thinking. All participants' interviews were audio recorded and the teachers reflected on how they used the visual tools to construct models to facilitate the lessons to enhance mathematical thinking. This data helped me to answer research question 3.

In this phase, I engaged with the participants by conducting a one-on-one stimulus recall interviews (Fox-Turnbull, 2009) with each teacher after their lesson presentations that took place in the previous four months. The purpose of the interviews enabled me (the researcher) and the participants to discuss the details of the lessons. The interviews were also used for participants to express their experiences on visual teaching strategies to enhance mathematical thinking in Grade 11 probability lessons. The questions for these interviews were directly related to the three research questions (see Appendix I). The interviews were recorded and transcribed. This helped to answer research question 3.

3.7 DATA COLLECTION TECHNIQUES AND INSTRUMENTS

In this study I employed the following techniques to generate data: questionnaires, lesson observations, interviews, and reflective journals. The purpose was to use multiple data collection techniques to triangulate the data. One of the reasons for utilising these three instruments together, was to make up for the deficiencies in each one (a process called triangulation) (Bertram & Christiansen, 2014), and put more distinct data forward. Table 3.3 below presents the different data collection methods used to collect data for the study.

Table 3.3: Data collection methods and process.

Phases	Activities in each phase	Data sources	Research question to be answered	Instruments	Data analysis
1	A survey in the region	Qualitative data	1	questionnaire	Stage 1
2	Selection of site and participants	no data			
3	• **Workshop 1: induction** Brief about the study • Discussion on topic objectives and how to teach probability. **Workshop 2: Planning** • Discussion on topics • Preparing lessons • Materials identification.	Qualitative data Qualitative data		Reflective journal Reflective journal	
4	Implementation (teaching) Observation Video-recording Field notes	Lesson observation Field notes	2	Observation schedule Reflective journal Field notes	Stage 2 Analytical tool
5	Interviews	Interview transcripts	3	Interview schedule	Stage 3 Reflective journal & interviews

3.7.1 Survey questionnaire

As mentioned earlier, it was important for me to understand how teachers of secondary school mathematics teach probability in the Khomas Region before I embarked on an intervention programme. For me to gain adequate understanding of this phenomenon, a questionnaire was used to collect data from the teachers. Bertram and Christiansen (2015) explain that a questionnaire is a list of either closed-ended or open-ended questions which the respondents answer. Also, it enables the researcher to standardise the questions asked, as highlighted by (Bertram & Christiansen, 2015).

I developed a questionnaire which consists of two sections – A and B – whereby Section A is about the participant's general information while Section B questions refer to the teaching of probability.

In section B, there were ten questions in total, of which eight required the participants to explain in detail how they teach probability using visual tools. The last two questions of Section B required the participants to tick the level of which they agreed to each item: 'never', 'some lessons', 'most lessons', and 'every lesson'. As mentioned, I only analysed thirty-four questionnaires out of the fifty questionnaires distributed in the Khomas region.

Furthermore, the questionnaires helped me to obtain data about how teachers in the Khomas region taught probability for promoting mathematical thinking prior to the intervention programme. The questionnaires further enabled me to get information on the teachers' backgrounds, their teaching methods and approaches and the types of visuals they used. Through the type of visuals used, I got information on the teachers' experiences and whether using visuals was linked to their new way of teaching after the intervention programme.

In my study, I took cognisance of the fact that completing questionnaires might be a challenge to some participants. To facilitate the completion of questionnaires, I first explained the questions to the participants during the induction workshop so that I could make things clear to them. Additionally, I included my cell phone number on the questionnaire so that the participants could conduct me if they faced any difficulties. Out of a total number of fifty questionnaires distributed to the participants, thirty-four questionnaires (68%) were retrieved. Perhaps this was caused by the fact that schools were closed unexpectedly because of the Covid-19 pandemic that struck the world from which Namibia was not excluded. Therefore, not all the participants handed back their questionnaires.

3.7.2 Lesson observation

It has been argued that observation is a powerful tool for gaining insight into a situation, although it has the challenge of being potentially intrusive (Bertram & Christiansen, 2015). For instance, the presence of the researcher in the learning environment may make both learners and the presenter behave differently. This effect is known as the Hawthorne effect (Oswald, Sherratt, & Smith, 2014).

Bertram and Christiansen (2014) suggest that observation enables the researcher to report on things she witnessed and recorded herself as opposed to things other people have to tell her. In my study I also observed and video recorded eight lessons in total. During the observations I sat at the back of the classrooms from where I video recorded all the lessons. I therefore made sure that all the participants understood my role as a researcher and encouraged free participation during lessons. Learners were actively involved in the lesson's discussions, as well as in taking turns to do hands-on practical activities, I was convinced as a researcher that my presence did not influence their participation.

Doing the observations was an exciting experience for me as a researcher, the presenter and the learners. I also instructed the participants to keep writing journal reflections on the presentations and of course an overall reflection after all presentations were done. I used stimulated-recall interviews to complement and triangulate data generated from the observations.

3.7.3 Stimulated-recall interviews

An interview may be defined as a conversation between the researcher and the respondent. Moreover, a stimulated-recall interview (SRI) is a collaborative inquiry between research participants and the researcher, through video or audio recall, with the dialogue focused on practice (Lyle, 2003; Nguyen, McFadden, Tangen, & Beutel, 2013). Additionally, SRI is known to have a potential strength as a clear professional development tool, whereby teachers can critically reflect on their own teaching practice (Reitano, 2005).

I conducted one-on-one stimulus recall interviews after the teachers had presented all their lessons which I had observed. The participants were allowed to check the videos of their lessons before the interviews started. This was done to help the teachers to reflect easily on what they what they had taught. The interview questions were guided by the themes generated from interview questions and the questions were directly related to how participants used visuals in their lessons. My focus was on the teachers' views and experiences of using visuals in teaching probability to promote mathematical thinking.

After every three lessons presented, I conducted an SRI with each of the three participants. During the SRI, I first played a recorded video of the presentation for the teachers to recall and refresh their ideas and thinking as suggested by (Cohen et al., 2017). The interview focused on the teachers' experiences of using visual probability teaching strategies to develop mathematical thinking within the Grade 11 learners. Table 3.4 below shows the duration of the interview.

Table 3.4 shows the durations of each participant's stimulus-recall interview.

Name of the teacher	Stimulus-recall interview duration	Interview code
Mr Assery	28 minutes	ASRI
Ms Emanu	17 minutes	ESRI
Ms Silvy	26 minutes	SSRI

There was a challenge because some teachers could not answer the questions in detail as I expected. This could be because they perhaps needed more time to prepare themselves after watching the videos.

3.7.4 Teachers' reflective, journals

A reflective journal is a place to write down your daily reflection entries. It can be something good or bad that happened to you that you can self-reflect and learn from experience McMillan and Schumacher (2014) explain that journals are a personal account of the learning experience. It is a way of allowing teachers an opportunity to express their feelings and views about the teaching experience so that they can make suggestions. As a strategy to prepare the participants for writing reflections, I made it clear to them during the induction workshop that they would be writing reflective journals during the intervention programme as well as after the presentations of the planned lessons. The reflective journal data and field notes helped to shape the interview questions.

3.8 DATA ANALYSIS

The data analysis of the study took place in three stages. Analysis is a close or systematic study (Bertram & Christiansen, 2014). Data analysis is a "process of making sense out of data" (Merriam, 1998, p. 178). This study used a mixed-methods approach which enabled me to collect and analyse the data in two ways: qualitative and quantitative data from the survey questionnaires, followed by qualitative data from the lesson observations and interviews during the intervention as shown in Table 3.5.

Table 3.5 Below shows the summary of the data analysis process.

Stages	Data source	Analysis
Stage 1	Survey questionnaires	• Analysed the questionnaires to find out the types of visual tools used by the mathematics teachers in the Khomas region. • Analysed the teachers' views on how to teach probability using the visual tools
Stage 2	Observations	• Transcribed videos taken during observation. • Analysed the transcripts using the visual probability teaching strategies analytical tools and mathematical thinking analytical tool. • Identified emerging themes analysed.
Stage 3	Stimulus recall interviews	• Transcribed the audio recording. • Analysed the interviews using thematic analysis

3.8.1 Stage 1: Survey questionnaires

This study collected both quantitative and qualitative data. Quantitative data were analysed by first using descriptive statistics to explore the teaching methods including 'when' and 'why' they chose those methods for various probability concepts. I further analysed data as to what type of manipulatives, how, why, and when they were used by the teachers in Khomas. Thereafter, data were presented in frequency tables and bar graphs.

The qualitative data from the questionnaires were analysed using the themes generated from the questionnaire questions. Those themes are as follows: ***teacher's training; visual teaching of probability; probability teaching in secondary schools; visual materials for teaching probability; sources of teaching materials; the use of visual tools and learners' understanding; visual tools used and mathematical thinking exercised; the importance of using visual materials in teaching probability; teachers' ways of improving the teaching of probability; and applying probability in reality.***

3.8.2 Stage 2: Observations

In this stage, the quantitative data from lesson observations were analysed. The first focus in this data analysis was on the visual teaching strategies used by the selected teachers to teach probability, with particular attention to mathematical models constructed. I therefore gathered all the teachers' excerpts as per the theme, from the two analytical tools and presented them on the tables. Initially, I presented all the excerpts from lessons that were related to visual probability teaching strategy themes on the table (see Chapter 4). Interestingly, there were other visual processes that emerged, such as the use of external representations. The emergent theme was therefore ‘the external visual presentation in probability lessons’. I also presented the teachers’ excerpts for the emergent theme on a different table.

Lastly, I focused on the how mathematical methods were used to enhance mathematical thinking. This was observed through mathematical thinking indicators as seen on the Analytical Tool B (see Appendix H). This is where I looked for instance at how teachers build on the learners’ prior knowledge, represent mathematical concepts differently, extend probability to real-life contexts and how teachers focus on structural methods and procedures of mathematics. I then continued to present all the teachers’ excerpts related to mathematical thinking and presented them on another table. The analytical tools were used to analyse the qualitative data from classroom observations. The Analytical Tool A helped to analyse the process of the use of visuals to teach probability, while Analytical Tool B helped to trace how the use of visuals enhanced mathematical thinking in the Grade 11 learners.

3.8.3 Stage 3: Interviews

The qualitative data generated from the interviews were transcribed and analysed thematically. Thematic analysis is defined by Braun and Clark (2006) as a method of identifying, analysing and reporting patterns within the data. Therefore, in this study the thematic analysis reported the experiences, views and realities of the participants. The thematic analysis method inspects the ways in which the meanings and realities of the lessons were understood by the teachers. The teachers’ responses were therefore transcribed and analysed as per the emergent themes from the interview questions, to categorise the similarities.

The emergent themes used for thematic analysis are outlined as follows: **selection or preparation for visual tools to teach probability; the importance of using visual tools; types of visual tools used by the teachers; experience gained from working with visual tools to enhance mathematical thinking; challenges or limitations in the use of visual tools; and the new way of teaching probability after the intervention programme.**

3.9 RELIABILITY AND VALIDITY

Validity refers to the degree to which a test measures what it is supposed to measure. According to Maxwell (2008) validity is the key issue in the debate over the legitimacy of qualitative research. However, the standard for rigour in qualitative research differs from that of quantitative research (Merriam, 2009). Franklin and Ballan (2001) note that issues related to validity and reliability might differ based on the nature of research conducted and the philosophical and ontological assumptions of the researcher.

In this study, I employed multiples strategies to ensure and enhance the validity and reliability of data. The multiple strategies included triangulation as I used several methods of data collection such as questionnaire, lesson observations, interviews and reflective journal that allowed me to trace the consistency in the participants' responses. I also piloted the questionnaire to the Grade 10 teachers to check whether the questions were appropriate and clear to the participants. I further piloted the lesson to check the use of video recording instruments and the analytical tools. The member checking method was also used as I gave the participants opportunity to go through the lesson observation and interviews transcripts to ascertain whether I have captured the data accurately and whether there is anything that might cause discomfort. Similarly, Stimulus recall interviews validated the video recordings by replaying specific episodes which helped the participants to remember/ recall their actions on the use of visual tools when teaching probability. The collection of data from the lived and perceived experiences of the participants helped to validate the findings of this study as they provided their first-hand experience of the use of visual tools in the teaching of probability.

According to Franklin and Ballan (2001), validity must take on different meanings and use different techniques with qualitative research. McMillan and Schumacher (2001, p. 407) describe validity in qualitative designs as "the degree to which the interpretations and concept have mutual meanings between the participants and the researcher".

3. 10 ETHICS

This study honours the ethics for conducting an educational research in various steps. Firstly, as a requirement I obtained ethical clearance for this study from the Rhodes University Higher Degree committee (see Appendix A). My application was forwarded to the Rhodes University Human Ethical Standards Committee and thereafter, the clearance certificate was granted upon approval. Secondly, I obtained the consent from the Regional Director of Education and then the written consents from the three principals were obtained too. Thirdly, I sought written consent from research participants, parents of the learners who were taught and video recorded, and this included approval to use the data for publication in journals and conferences. Lastly, I explained the purpose, objectives and possible contribution for the study to the research participants and the school headmaster.

All participants were informed that they had a right to withdraw anytime from the study if they felt so. Pseudonyms were used so that the participants' names would remain anonymous. The participants for the survey questionnaires were referred to as $T_{1,}$ $T_{2,}$ and so on, while the teachers that took part in the intervention were given pseudonyms and referred to as Mr Assery, Ms Emanu and the last one as Ms Silvy. As observation data were collected by means of video and audio recordings, participants were regularly reminded that their faces would be kept hidden. There was a high degree of respect between me as a researcher and the participants.

During this time, when the countries were trying to curb the spread of COVID-19, schools changed their timetables so that the learners attended school on alternate days, all the non-promotion subjects including Information Communication Technology (ICT) lessons were removed from their timetables as per the directive from the Ministry of Education. Consequently, the teachers and senior learners had more free period. Therefore, our lessons were scheduled to be taught during these free periods. This helped to occupy the learners instead of being unsupervised during free lessons. However, not all their free lessons were utilised. Some free periods were used for homework, projects and for studying as they awaited the external examinations. Therefore, the teaching for this research study did not disrupt the normal programme of the school.
For transparency's sake, I ensured that my participants were involved in every phase of the study by giving them opportunities to verify their responses in SRI. All my participants were my fellow teachers and I thus engaged them with integrity at all times, keeping in mind the ethical standards of Rhodes University. Integrity and professionalism were observed in this study at all times. I also acknowledged and referenced other authors' work according to the Rhodes University referencing guide for academic writing. I am glad to report that this entire study is my own work under the guidance of a very supportive supervisor.

3.11 CONCLUSION

This chapter gave a detailed description of the research paradigm that guided the research design and methodology. It further discussed the survey and the five phases that I used in my research design to enable me to answer the research questions. The dominant instruments for data collection were the survey questionnaires, observations and interviews. The chapter further looked at the data analysis and the sampling process of my study. Lastly, I presented the validity and ethical aspects of the study. The next chapter presents data collected from the data collection instruments discussed earlier in this chapter. Therefore, in the next chapter, I analyse, present, and discuss and interpret the data generated in this study.

CHAPTER 4
DATA ANALYSIS, PRESENTATION, AND DISCUSSION

4.1 INTRODUCTION

In this chapter, I report on the findings of my study that sought to explore visual probability teaching strategies for enhancing mathematical thinking. The goals of the research were firstly: to find out the probability teaching strategies used by secondary teachers in the Khomas region to promote mathematical thinking; secondly, to establish the extent to which the selected teachers use visualisation tools to promote mathematical thinking during the teaching of probability in Grade 11 classrooms as a result of participating in an intervention programme; and thirdly, to explore and gain insights into the teachers' experiences of using visualisation tools to teach probability for promoting mathematical thinking, as a result of an intervention programme. The findings provide the necessary information and argumentation to answer the three research questions that guided this study, which are:

1. How did secondary school mathematics teachers in the Khomas region teach probability to promote mathematical thinking before the intervention programme?
2. How do selected teachers use visualisation tools to promote mathematical thinking during the teaching of probability concepts in a Grade 11 classroom, as a result of participating in an intervention programme?
3. What are the selected teachers' experiences of using visualisation tools to teach probability for promoting mathematical thinking, as a result of an intervention programme?

In the next section, I start by giving a short description of the how I designed the analysis of the data. Survey data were analyzed quantitatively using descriptive statistics. I then give a brief overall description of the analytical tools that were used to analyze the second and third research questions.

4.2 SURVEY DATA ANALYSIS

The survey was designed to provide the basis for the whole study and specifically to answer the first research question (see Section 4.1). The survey was also used to find the teachers' views on how they teach probability in their classrooms. Thirty- four mathematics teachers ($n = 34$) in the Khomas region responded to the survey questionnaire. For easy identification of respondents, questionnaires were labeled from Teacher 1 (T_1) to Teacher 34 (T_{34}). The teachers' responses helped to craft the intervention programme.

4.2.1 Teaching experience

Table 4.1 below gives an indication of the number of years of experience the teachers in this study had.

Table 4.1: Teaching experience of respondents

Teaching experience (years)	≤ 5 years	6-10	11-15	16-20	21+
Frequency percentage	32%	41%	15%	12%	0%

Table 4.1 shows that 32% of the respondents to the questionnaire had five or fewer years of teaching experience while 68% had more than five years. This was of interest to this study as I consider five years as adequate experience for a teacher to have experimented with various approaches and to gain sufficient practical knowledge of teaching and learning. It was also interesting to note that 27% (see Table 4.1) of the teachers who took part in this survey had over ten years of teaching experience. This helped me to start thinking of how I was going to structure an intervention programme from which such highly experienced teachers would benefit.

4.2.2 Teacher Qualifications

Table 4.2 below shows the qualifications of the teachers who responded to the survey questionnaire.

Table 4.2: Teachers' qualifications

Teachers' Qualifications	Grade 12	Certificate	Diploma	Undergraduate degree in education	Honors degree in education	Masters in education
Frequency %	0	0	18	35	44	3

The questionnaire needed to target qualified teachers in the Khomas region's schools. Table 4.2 shows that of the thirty-four respondents, 18% had a Diploma in Education while 82% had at least an undergraduate degree. As indicated in Table 4.2 above, there was no unqualified teacher that responded to this questionnaire. This made me anticipate that the qualified teachers would have at least some tangible ideas, better knowledge, and be in a better position to share their probability teaching approaches using visuals to promote mathematical thinking. This could also help me as a researcher to find the rich data for the study.

4.2.3 Teachers' training

This section probed whether the teachers had covered the topic of probability during their university training and the role of visualisation in mathematics. The respondents indicated that they had covered the topic of probability during their training and that the role of visualisation in mathematics was clearly explained to them during their training. Table 4.3 below indicates their responses to the survey question regarding their training in teaching probability and using visualization as a teaching strategy.

Table 4.3: The teachers' training on probability and visualisation

Teachers	***Responses from teachers***
T_1	*Topic covered during teacher training, the role of visualisation is understood as to show graphical data that help the viewer to see the displayed data and process it.*
T_3	*Topic covered, we were encouraged to always think of how he/she can use visualisation when presenting lessons as it is believed to improve learner's performance in mathematics and draw most learner's interest in the subject/topic.*
T_8	*Topic covered, visualisation in mathematics helps learners to solve problems, promote mathematics thinking, and for better understanding.*
T_{12}	*Topic covered during the teacher training and the role of visualisation in mathematics is seen as a common metaphor for understanding (learners being able to see something mentally).*
T_{13}	*Topic covered, visualisation seemed to enhance teaching and learning. This helps learners to remember well when they see and touch concrete objects.*
T_{26}	*Topic covered with little emphasis on probability.*
T_{27}	*Topic covered, nothing was covered on the role of visualisation in mathematics. It was a mere calculation using formulae.*

Teachers in this survey highlighted that they were encouraged by their lecturers to always think of how they can use visualisation when presenting lessons as it is believed to improve learners' performance in mathematics and draw learners' interest most successfully in the subject/topic. The respondents were able to note the advantages that visualisation brings in mathematics. These advantages included the ability for a visual approach to help learners to solve problems, think mathematically and gain a deeper understanding. This is also supported by T_{12} as is indicated in Table 4.3, that visualisation in mathematics is seen as a common metaphor for understanding (learners being able to see something mentally).

4.2.4 The visual teaching of probability

The respondents indicated that to illustrate the concept of probability in a visual manner, they would bring to the lesson real-life objects that are familiar to their learners. Teachers gave examples of objects such as coins, dice, beads, smart boards, spinners, flashcards, marbles and dart boards, among others (see T_2 and T_5 in Table 4.4). Table 4.4 below shows the teachers suggestions for teaching probability using visuals.

Table 4.4: Teaching probability in a visual manner

Teachers	Responses from teachers
T_2	*I believed that visuals can be in different forms whereby, one can use posters explaining the concepts of probability or using real-life examples. However, one can also use readily available materials like dice, cards, flipping of coins, and explain the chances of getting a certain occurrence.*
T_5	*I would visually illustrate the concept of probability by manipulating concrete and tangible objects such as beads and marbles to foster understanding to the learners.*
T_{11}	*I would use technology that will allow him/her to easily demonstrate concepts visually yet engaging learners. One can also make use of Venn diagrams and tree diagrams to represent probabilities visually, drawing cards randomly from a deck of ten cards.*
T_{14}	*I would play YouTube videos and demonstrate using dice.*
T_{25}	*Provide external visual representations that may aid scientific problem-solving e.g. diagrams, tree diagrams, contingency, possibility diagrams tables, Venn diagrams, and pictures.* *… formulae and mathematical symbols could be included in the list because they incorporate visuospatial relationships.*

Respondents indicated that teachers need to demonstrate and show the learners to foster understanding during learning (see T_5 in Table 4.4). In the same vein, respondents further stated that the use of technology when teaching probability e.g. YouTube videos (see T_{11} and T_{14} in Table 4.4) is vital to easily demonstrate the concept visually yet engage the learners to learn probability through experiences of critical thinking.

Other respondents believed that visuals can be in different forms (see T_2 in Table 4.4), whereby one can use posters explaining the concept of probability. They further highlighted that one can also make use of Venn diagrams, possibility diagrams and Tree diagrams to represent probabilities visually (see T_{11} and $T_{25,}$ in Table 4.4). Table 4.4 further indicates that formulae and mathematical symbols could be included in the list because they incorporate visuospatial relationships (see T_{25}). Among the respondents, it is noted that some teachers did not comment on how they would visually illustrate probability, as they seemed to have no ideas. By looking at the teachers' responses it was very interesting to note that the teachers are aware of various visual tools that can be used to teach probability for promoting mathematical thinking as mentioned in Table 4.4 above. However, a few of the respondents left the question blank or unanswered, indicating that they have no idea how they would illustrate probability visually.

4.2.5 Probability teaching in secondary schools

Table 4.5 below shows that respondents indicated that they would teach probability by defining the topic, giving practical examples, and demonstrate using readily available tools e.g. coins, beads, dice, cards, spinners, graphs, charts and pictures, as well as giving real-life examples (see T_2, T_{11}, and T_{15} in Table 4.5).

Table 4.5: Probability teaching in the classroom.

Teachers	Responses from teachers
T_2	*I would explain the concept of probability (the chance of something to happen out of the total chances). This would be done by using real objects or real-life examples like if you write exams, there are two (total) chances, it is either you pass or fail to ask the learners the probability to fail/pass.*
T_3	*I would explain the concept of probability by giving more examples and prepare a short presentation or show the learners stimulations which will make the learners understand even better.*
T_{11}	*Defining probability and ensure that learners understood the concept. Make use of the available resources and clear the misconceptions associated with probability. Use a variety of engaging activities where learners participate actively. Give learners time to think, raise questions that provoke mathematical thinking, and ask for learner's explanations and reasoning.*
T_{15}	*… by starting with simple problems and define the complex ones. Make use of models to explain and make use of drawings like tree and Venn diagrams etc on the chalkboard.*
T_{16}	*… by playing a probability game with the whole class.*

It is further noted that one would start with simple problems and progress to complex ones. Therefore, most teachers encouraged the use of models in classrooms and allow learners to practice and engage in critical thinking (see T_{15} and T_{16}). It was also interesting to note that the respondents highlighted the need for the teacher to facilitate the lesson, raise questions that provoke mathematical thinking, give learners enough time to think, and ask learners explanations for their answers and reasoning (see T_{11} in Table 4.5).

Similarly, respondents also showed the need to play probability games with the whole class by allowing learners to practice with the available tools and engage in teacher-learner interactions (see T_{16} in Table 4.5). T_3 encouraged the use of stimulations to teach probability. Few respondents encouraged the use of graphs, Venn diagrams, possibility diagrams, and Tree diagrams. Furthermore, they highlighted that they would teach probability by demonstrations, discussions with learners, and invite learners to the chalkboard to do selected activities while the teacher facilitates the lesson.

Additionally, other respondents indicated that they would teach probability by calculating percentages while other respondents had no idea or did not comment on this question. Both these two types of respondents seemed to have no clue on how to teach this topic effectively, therefore appropriate measures have to be put in place to help such teachers. In the same vein, as much as the teachers acknowledged that it is significant to use models in classrooms and allow learners to practice and engage in critical thinking, a few respondents highlighted that they would teach probability by calculating percentages while others had no comment. Both of these participants made me as a researcher start thinking about how can they be helped to teach probability effectively while promoting mathematical thinking.

4.2.6 Visual materials for teaching probability

Respondents acknowledged the use of visual tools for enhancing mathematical thinking in probability classrooms. These included real objects like dice, coins, highlighters, cards, coloured balls, beads, spinners, sharpeners, marbles, flashcards, packs of crayons, coloured chalks, etc. (see T_1 in Table 4.6). Apart from the concrete objects, respondents also highlighted the need to use posters, drawn Tree diagrams, Venn diagrams, and possibility diagrams to mention but a few (see T_8 in Table 4.6). Table 4.6 below indicates that they also use presentations, stimulations, and YouTube videos for similar lessons. (see T_3 and T_{13}).

Table 4.6: Visual materials for teaching probability

Teachers	Responses from teachers
T_1	*Real objects like dice, beads, stones, containers (buckets), and any other relevant readily available materials.*
T_3	*Presentations, stimulations, colorful beads.*
T_8	*Graphs, charts, Coins, pictures, tree diagrams, contingency tables, Posters with Venn diagrams, and tree diagrams, etc.*
T_{13}	*YouTube videos for similar lessons*

It was interesting to note that teachers are aware of other visuals used to teach probability apart from concrete objects used in classrooms like dice, coins, flashcards, etc. Teachers acknowledged the use of presentations, stimulations, posters with Venn diagrams and Tree diagrams as well as YouTube videos for similar lessons. Conversely, some respondents showed that they used a chalkboard as visual material to teach probability. This showed a lack of understanding of what exactly visual tools represent. Other respondents had no comment, therefore, it seemed that they had withdrawn from the research because they left questions unanswered.

4.2.7 Sources of teaching materials

Table 4.7 below shows that the teachers either collect/get their teaching materials from their houses, create or develop them (spinners), or buy them from nearby shops using their own finances. However, T_5 indicated that they also collect some teaching materials from the school like coloured chalks, sharpeners, highlighters, crayons, etc. to mention but a few.

Table 4.7: Sources of teaching materials

Teachers	Responses from teachers
T_2	*Buying materials and prepare the visual tools myself, e.g. flashcards, spinners, etc.*
T_3	*Buying cards, download the stimulation from the internet and prepare own materials.*
T_5	*Buying from local shops from personal finances improvise from available materials. learners can also prepare them.*
T_{14}	*From magazines or the internet and bought from personal finances*

Furthermore, it is also noted that the respondents acknowledged the use of magazine extracts as well as downloading simulations from the internet. Again, T_{19} did not comment on that question. By looking at the teachers' responses in Table 4.7 above, one can infer that not all teachers buy visual tools for teaching probability but that teachers should be creative enough to use any readily available materials from their houses or school surroundings.

4.2.8 The use of visual tools and learners' understanding

As the saying goes "seeing is believing", the respondents believed that the use of visual tools helped learners to understand probability as they are able to see how ideas are connected, leading to a thorough and easy understanding of new concepts when they experience concrete examples (see T_1, T_{18}, and T_{12} in Table 4.8). It is noticed that most of the learners are visual learners and pay more attention to presentations and stimulation videos compared to learning theory (see T_3 in Table 4.8). Table 4.8 below shows how teachers see the use of visual tools having an effect on learners' understanding.

Table 4.8: Use of visual tools and learners' understanding.

Teachers	Responses from teachers
T_1	*It helps learners to visualise (see and touch the object). Many learners learn better by seeing than hearing especially when the teacher is using familiar material (real-life examples). Project a spinning wheel or prepare one.* *Visual tools arouse and stimulate learner's curiosity and inquisitiveness. It is very easy for them to relate and work out.*
T_3	*I noticed that most learners are visual learners and pay more attention to presentations and simulation videos comparing to when the teacher is explaining.*
T_{12}	*Yes, learners see how ideas are connected leading to a thorough and easy understanding of new concepts as they experience concrete examples. Pictures help learners see mathematical ideas practically which facilitates higher-level thinking.* *-what the child participates in and has a first-hand experience, they remember easily, though it does not mean it's absolute.*
T_{18}	*Most learners learn easily by seeing. Visual tools aid better comprehension i.e. visuals help learners grasp concepts easily by stimulating imagination and affecting their cognitive capability.*
T_{22}	*Yes, it has been effective for three years now, learners understand better. This allows them to engage more in critical thinking that sharpens their mathematical thinking.*
T_{25}	*Yes, it is an eye-opener* *… it gives confidence* *… helps in developing the love and interest for the subject that leads learners to practice regularly on the concept.* *Make learners enjoy the topic.*

It was also interesting to notice that visual tools are helpful and aid in solving problems, and arouse and stimulate learners' curiosity and inquisitiveness (See T_1 in Table 4.8). It also makes the learners happy e.g. by playing probability games in the classroom like cards, spinners, rolling dice, flipping coins, etc. The mind tends to remember things when they are done physically. Furthermore, T_{25} also added that visual tools are the eye-opener that give confidence and help in developing love for and interest in the subject or topic. Similarly, T_{22} acknowledged that the use of visuals has been effective for three years now in his classroom (see Table 4.8) since she/he started teaching using visuals. This encourages the learners to enjoy the concept/topic and engage more in the critical thinking that sharpens their mathematical thinking (T_{22}). As a saying "A picture is worth a thousand words" goes, these respondents believed that seeing something is better for learning than having it described (theory). It was noticed that the majority of the teachers were speaking with one voice; they acknowledged that the use of visuals promotes learners' understanding as they experience concrete examples. They further highlighted that visual tools are the eye-opener that give confidence and help in developing a love for and interest in the subject or topic.

4.2.9 Visual tools used and mathematical thinking exercised

Table 4.9 below shows that visual tools provoke reality and learners engage in practical thinking rather than theoretical thinking (see T_{24}). It is noted that learners easily remember what they have seen and continue to actively think further. On the same note, it is indicated that learners learn in different ways, and therefore it is of great importance for teachers to apply varied instruction when teaching to cater for all the learners (see T_1). They further highlighted that visual tools increase learners' engagement and participation e.g. throwing dice or flipping a coin increases learners' participation and learning. This might also increase motivation and confidence as well as changing the learners' mindset. Learners, therefore, develop higher thinking order skills in manipulating visual tools which enhance their mathematical thinking.

Table 4.9 Visual tools and mathematical thinking

Teachers	**Responses from teachers**
T 1	*Learner learns in different ways. Some learn best by seeing, so a teacher must apply different instructions when teaching mathematics.* *Learners will develop higher thinking order skills in manipulating visual tools which will enhance their mathematical thinking.*
T_3	*Learners are prompted to think critically especially when watching stimulating (ion) videos. The teacher pauses the video after a problem or question was posed to allow them to think of the answer first before it is revealed.*
T_{11}	*It can be a powerful cognitive tool in problem-solving.* *It allows learners to identify a simpler version of the problem.* *It allows learners to solve problems* *It allows the learners to formalize the understanding of the given problem.* *Allows learners to identify a method that will work for all similar problems i.e. relating the given problem to previous problem-solving experiences.*
T_{12}	*The use of visual tools develops learners' critical thinking skills.* *Deepens understanding, allows learners to debate with confidence and predicting feature events, interpret situations,* *It creates a context that justifies that mathematics can be applied in real-life but not just an end in itself.*
T_{24}	*I use visual tools that provoked reality.* *Learners engaged in practical thinking than theoretical thinking. It makes learners think critically and connect it to their real-life situations.* *Dice help learners improve their mathematics skills through math mental games.*

The respondents further highlighted that visual tools can be a powerful cognitive tool in problem-solving where they allow learners to identify a simpler version of the problem (see T_{11}), allow them to solve the problem, to formalise their understanding of the given problem and identify a method that will work for all similar problems (i.e. relating the given problem to previous problem-solving experiences). T_{12} confirmed that the use of visual tools develops learners' critical thinking skills and create content that justifies that mathematics can be applied in real-life but not just as an end in itself – thus connecting theory to reality. Additionally, it was noticed that when teachers use visual tools, learners tend to perform better. Therefore, visual tools like dice help learners improve their mathematics skills through math mental games (see T_{24} in Table 4.9). Pictures help learners see mathematical ideas which help to understand visual mathematics, create a higher level of thinking, enable communication and help people see the beauty in mathematics.

4.2.10 The importance of using visual materials in teaching probability

There is great significance in using visual materials in teaching probabilities since the topic seems to be quite challenging (see T_5 in Table 4.10). Table 4.10 shows that respondents understood the importance of using visualisation tools as follows: Firstly, learners learn in different ways and some learn best by seeing (see T_1). Therefore, teachers must use/apply different methodologies when teaching mathematics. Secondly, it was also noted that visual tools help to develop high-order thinking skills and enhance tactile hand-eye-mind connections that improve the ability to recall facts and retain probability skills (see T_{12} in Table 4.10). Lastly, visual tools seem to help learners to understand the definition of probability, which may lead to conceptual understanding and procedural fluency. T_{18} noticed that visual materials encourage the learning process and make it easier for the teacher to explain the concept. Therefore, it is advocated that the use of visual tools in teaching probability is the best dissemination of knowledge (see T_{18}). Table 4.10 below shows the teachers' responses to the importance of using visual tools when teaching probability.

Table 4.10: The importance of using visual materials in teaching probability.

Teachers	Responses from teachers
T_1	*Learners learn in different ways some learn best by seeing.* *I must use/apply different methodologies when teaching mathematics.* *Probability is taught better through experiential learning "learning by doing" which assists learners to understand the concept better.* *To enhance learner's mathematical thinking.*
T_2	*It increases learner's participation, motivation, engagement, confidence, and improves the mindset.* *It will help learners to remember the concept faster and deepens their understanding. It motivates learner-centered as learners will be involved more in discussions etc.*
T_5	*There is a great significance in using visual materials in the teaching of probabilities since the topic is quite challenging.* *Using visual tools fosters understanding.* *Probability is applied in real life.* *Helps learners to understand the concept, improve their lives through making better life choices such as the probability of car accidents, etc.*
T_{12}	*It develops high-order thinking skills.* *It enhances tactile hand-eye-mind connections that improve the ability to recall facts and retain probability skills.* *Creates the love and interest for the subject.* *allows learners to predict and make wise decisions in the future.* *Makes the lesson live.*
T_{18}	*Encourages learner's learning process and makes it easier and interesting.* *Makes teaching effective, opens the learner's mind. It is the best dissemination of knowledge.*

Table 4.10 further shows that visual tools further arouse interest and curiosity, increase learners' participation, motivation and engagement, and change their mindsets for the better (see T_2). It is also noticed that visual tools motivate learner-centeredness as learners are more involved in discussions, and encourage critical thinking promoting better understand and mathematical thinking. In the same vein, the use of visual tools encourages fast learning, better understanding, and quick thinking. Furthermore, these respondents further highlighted that visual materials are seen as models that deepen learners' understanding, make the lesson live, give confidence and create a love for the subject (see T_{12} in Table 4.10). Most respondents believed that as much as visual materials deepen understanding, they also allow learners to predict and make wise decisions in the future.

A few respondents seemed to have no idea why it is vital to use visual tools to teach probability. This is shown as their answers did not correspond to the question. (Their answers were *"to ease retention"*, *"visuals are important because the kids are also important"* and the rest had no comment). There seems to be a need for a refresher course as some teachers who have been in the field for a long time have perhaps become set in their traditional ‘chalk and talk’ ways of teaching.

4.2.11 Teachers’ ways of improving the teaching of probability

Respondents indicated that there is a need for teachers to incorporate more visual tools in their teaching (see T_{14} in Table 4.11). More importantly, teachers need to be self-motivated and motivate learners regularly. Teachers, therefore, acknowledged that learners need enough time to practice and think over the concept (see T_{30} in Table 4.11). They further added that teachers have to assess mathematical thinking regularly and be alert to opportunities to invite mathematical thinking. It is also interesting to note that teachers need to do the following for the effective probability lessons: ask learners to explain their methodologies; anticipate learners’ reactions and prepare to activate follow-up; pay attention to the problem-solving process/methods; have whole-class reflections on strategies; actively engage learners in constructive learning; and raise questions that will require critical thinking to enhance mathematical thinking (see T_{30} in Table 4.11). T_8 pointed out the need for teamwork as learners practice and learn from their peers on best practices. Table 4.11 shows what the survey respondents had to say regarding improving the teaching of probability.

Table 4.11: Teachers' ways of improving the teaching of probability.

Teachers	Responses from teachers
T_8	*The best way, work with probability articles* *Probability games will be helpful too* *Encourage teamwork and learn from their peers on the best practices.*
T_{14}	*Avail resources other than dice and coins* *Incorporate visual tools more, ICT, such as YouTube videos to compare different strategies and approaches rather than just doing examples from the textbooks, without bringing in the real-life applications of probability.* *Should give clear examples of how probability is used in daily lives, e.g. Corona situation*
T_{15}	*Use plenty of visual materials. Use different methodologies.* *Connect probability to the real-life situation (in most cases learners ask why should they learn probability.)*
T_{22}	*Increase the use of models in classes.* *Take learners to different companies to see practical work of probability.* *Offer training to improve professional practice because it is time to pressure teachers.* *Use of videos and other visuals to help better enhance learners' understanding.*
T_{30}	*I should actively engage learners in constructive learning.* *Allow learners to think by giving them enough time to practice.* *Raise questions that will require critical thinking to enhance mathematical thinking.* *Assess regularly and give feedback promptly.*

Even though it is noted that teachers should use external visual presentations to improve cognitive capability in learners, there is also a need for teachers to use resources other than dice, coins spinners or cards, etc. by incorporating ICT, such as YouTube videos to compare different strategies and approaches (see T_{14} and T_{22} in Table 4.11). Other respondents felt that the best way is to work with probability articles and probability games. It was also indicated by the respondents that it is very important for teachers to make lesson more lively and interesting by relating it to daily activities or trending news e.g. the spread of COVID -19. This may improve their teaching and help the learners to understand better (see T_8 and $_{14}$ in Table 4.11). It is believed that once learners have mastered the concept, they could go on to the extent of recreating their notes or discovering shorter methods in their imaginations. Teachers should, therefore, allow learners to figure out how to use mathematical tools to solve non-routine problems

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It is also noted that in some cases learners ask their teachers why they should learn probability. Therefore, there seems to be a need for teachers' training to improve professional practice so that they can handle daily probability problems. Similarly, learners need to be taken to different companies to see the practical work of probability (see T_{22} in Table 4.11).
A great concern was raised when some teachers responded that there is no need for teachers to improve the teaching of probability, yet the majority of learners are failing this topic. A few respondents had no comments on the way forward to improve the teaching of probability.

4.2.12 Applying probability in reality

Respondents pointed out that learners experienced a probability in their everyday life activities (see T_{26}) without understanding the actual mathematical concept on which such activities are based. Therefore, probability needs to be applied to reality by giving real-life examples of probability i.e. the likeliness of an event to occur and the impossibility of certain occurrences can be used to link the content to practice, for example, the idea of predicting the weather and rainfall, (weather forecast), playing a jackpot (gambling machine), betting for sports games, playing games where there is a 50/50 chance of succeeding/failing and using trending news and batting average in cricket, (see T_1, T_4, T_{12}, T_{18} and T_{22} in Table 4.12). Table 4.12 below shows how the respondents apply probability to reality in their lessons.

Table 4.12: Applying probability to reality (linking probability to real-life phenomena or daily arithmetic).

Teachers	Responses from teachers
T_1	*Using the idea of rainfall: i.e. When it is likely to rain due to the presence of clouds or we can say the probability that it will rain is zero due to the absence of clouds.* *Playing a jackpot, buying a car, coming to school naked, etc.* *Give an example of a gambling machine and explain how it works.* *Use the example of participation in competitions etc.*
T_4	*Use a practical example, the game of a darts board, tossing a coin, playing cards, politics, and dice.* *Use passing rate or gender probability.*
T_{12}	*It can be related to e.g. betting for sports games (English premium league) or playing games where there is a 50/50 chance of succeeding/failing.*
T_{18}	*Probability in real-life is used in batting average in cricket to explain the concept of probability.* *Probability is used in games and recreational activities e.g. (games that involve luck or chances).* *By using current trending news and place probability in that scenario e.g. COVID-19 situation.*
T_{22}	*Make use of models to help learners understand.* *Selling of rivalry tickets* *Entering competitions* *Make use of gambling, predicting the chances of spinners.*
T_{26}	*Learners experience probability in their everyday life activities without understanding the actual mathematical concept on which such activities are based. Spinning a spinner or rolling a dice (Classical probability situations) will help learners to interpret situations and make some predictions in real-life contexts.* *Allow learners to describe their thinking and understanding through the use of correct probability language.* *Give probability examples in reality e.g. probability of seeing/having sunrise at night.*

Moreover, it is therefore of paramount importance to help learners to interpret situations and make predictions in real-life contexts by allowing them to spin a spinner or roll a dice (classical probability situations) (see T_{26}). It is also noticed that some of the respondents had no idea how to link probability to real-life phenomena. Again, there seems to be a need for the teachers' workshops on probability.

4.2.13 Frequency of using visual tools

Figure 4.1 below shows the teachers' frequency of using visual tools in probability lessons. I noticed that some teachers were using visual tools in some, most, and in every lesson while others did not use visuals at all. Meanwhile, the teachers who used visuals did not use them in all stages of their lessons. This made me think that visuals are used on certain occasions where the teacher deems it necessary. Having said that, Ruzic and O'Connell (2001) proposed that manipulatives can be successfully used in introducing mathematics lessons, to practice, or to remediate mathematical concepts in mathematical instruction. Figure 4.1 shows that the highest record of teachers using visuals for reasoning in some lessons was 44%, and another 44% using visuals for working on problems without an obvious method in most lessons.

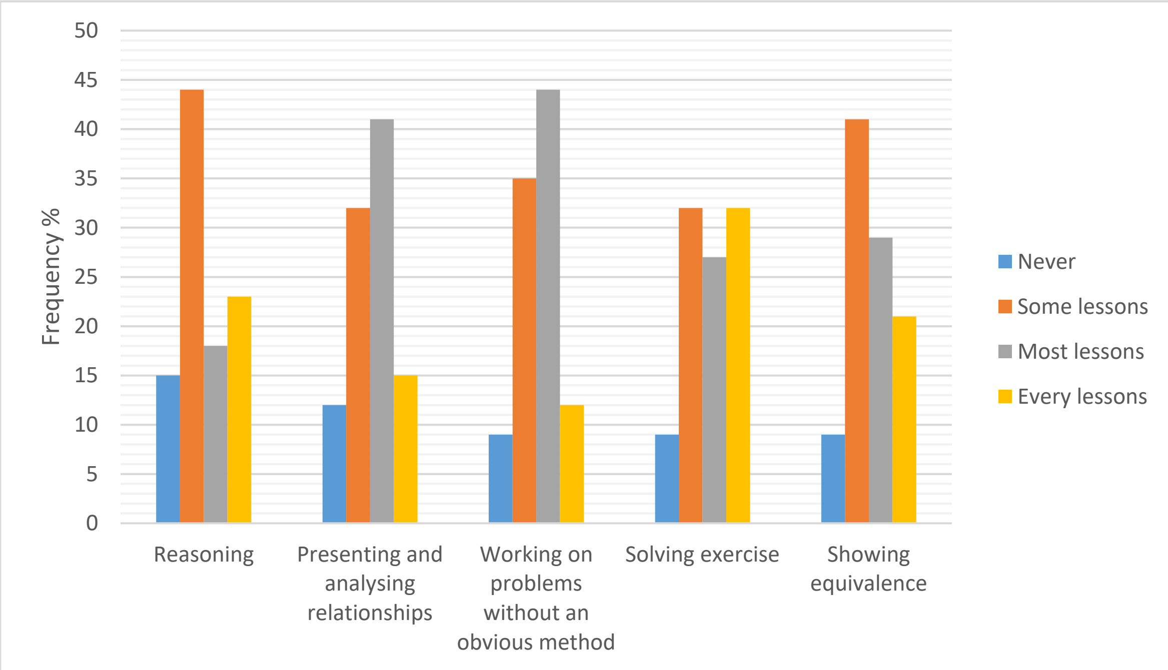


Figure 4.1: The teachers' frequency of using visual tools.

Using visual tools for reasoning and working on problems without an obvious method is not adequate because (as shown in Figure 4.1) they were only used in some lessons or most lessons. Another 41% of the teachers used visuals only to represent and analyze relationships in most lessons and showing equivalence in some lessons. Teachers need to use manipulatives to help learners to reason mathematically by aiding them to construct a meaningful understanding of abstract concepts (Cockett & Kilgour, 2015).

Interestingly, below 33% of the teachers used visuals for all the topics in every lesson. Hence this causes great concern about the teaching of probability which requires the use of visuals in both introduction and lesson presentation. Among others, there were also 32% of the teachers that used the visuals to solve exercises or problems in some lessons while the other 32% used visuals to solve problems or exercise in every lesson. This further indicates that at least the teachers were trying but it is crucial to improving the 'never' and 'some' responses to 'most' or 'every'. This showed that there was a need for the Khomas teachers to maximize the use of visuals in their lessons right from the introduction to lesson delivery to reinforce understanding of probability concepts.

4.2.14 Classroom interactions when using visuals

Figure 4.2 below shows the time frames for using visuals, according to the respondents. Teachers showed that learners worked: without the teacher's assistance in some lessons, with the teacher's assistance, together with the teacher, as a class, together responding to each other, or in pairs/groups without the teacher's assistance, and some teachers did not use visual tools at all. From all these incidents, one may deduce that the time frames for using visuals were not convincing enough because most of the percentages were below 40%, except for the learners working without teacher's assistance in some lessons, learners working with the teacher's assistance and learners working together with a teacher teaching the whole class. Figure 4.2 below shows the classroom interactions while using visuals.

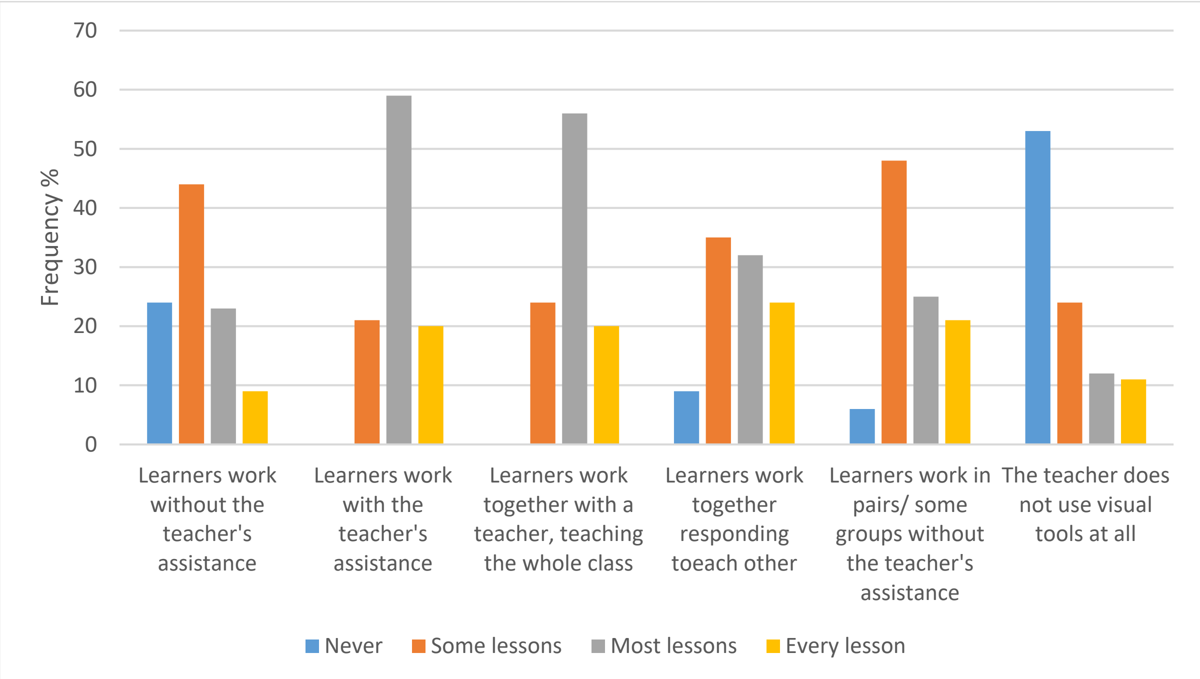


Figure 4.2: Classroom interactions when using visuals.

The surveyed teachers showed that learners work with the teacher's assistance in most lessons (59% in Figure 4.2) when using visuals. Moreover, 44% of the teachers further showed that learners work without the teacher's assistance in some lessons when using visual tools. This made me think that teachers need to be careful when handing manipulatives to the learners, to make sure that they serve the purpose they were planned for. Therefore, learners need to be monitored effectively to ensure that they are focused on the concept being taught. In support of this, Clements and Sarama (2016) suggested that although manipulatives can play an important role in the teaching of mathematics, they nevertheless need to be used carefully to create a strong understanding and justification for each step and procedure. Only

52% of my teachers revealed that learners work together with a teacher in most lessons, as a whole class using visuals. This further means there is still a need to incorporate visuals in more lessons because only 20% of the teachers revealed that they work with all the learners in every lesson. Hence, there was a need for intervention. The surveyed teachers' responses indicated to me the need for an intervention to assist teachers to help their learners use visuals more meaningfully on their own. Teachers would need to inculcate in their own learners the quest to work independently of the 'more knowledgeable other' (Vygotsky, 1978).

4.2.15 Summary of the survey

Generally, the teachers' responses in this survey indicated that visual tools and strategies were inadequately used to teach probability by the teachers in the Khomas Region. By considering teachers' responses, for example in Figure 4.1, teachers were using visual tools selectively, while there were those that did not use visual tools at all. Some respondents highlighted that the use of visual tools does not work best when learners work without a teacher's assistance (see Figure 4.2 for example), or when learners work together responding to each other and when working in pairs/groups without the teacher's assistance. This revealed to me that the teachers' understanding of visuals for teaching was limited.

Surveyed teachers who used some visuals did not use these visuals consistently in their teaching. Notably, their responses to the questionnaire showed that they appreciated the value and importance of using visualisation to teach probability. This was evidenced by the majority of the respondents who highlighted that the use of visual tools in teaching probability is the best way to disseminate knowledge. Their responses further pointed out that there is a great significance in using visuals since the topic (probability) is quite challenging. Therefore, visuals arouse interest and curiosity, develop high-order thinking as well as enhancing tactile hand-eye-mind connections that improve the ability to recall facts/concepts and retain probability skills. In addition, Ruzic and O'Connell (2001) suggested that using various manipulatives provides an exciting classroom environment, promotes positive learner attitudes toward mathematics learning and greatly reduces anxiety. Therefore, there seems to be a need for engaging teachers in several workshops as well as creating strong cluster bonds or engaging in co-teaching to share various skills.

Similarly, respondents further asserted that visual tools open learners' minds and help them to think deeper and faster. It is also noted that visuals improve the lessons by making them lively and help the teacher to explain the concept easier as the learners learn by doing (touching and seeing the real objects) rather than mere symbols or theory. Additionally, seeing and touching clears up misconceptions and confusion, and it leaves/prints a clear picture in the person's mind to enhance a broader and more realistic understanding. It is further argued that visuals make probability fun and practical as well as relating probability to real-life situations, thus enabling learners to connect probability learned in class to real-life situations. As a result, this prompted me to devise an intervention programme in which I was interested in fostering the use of visualisation tools or processes in teaching probability at secondary school, Grade 11 level.

4.3 FINDINGS FROM CLASSROOM OBSERVATIONS

This section presents the observations of the lessons taught by the three mathematics teachers from three different schools in the Khomas Region. Responses to the survey questionnaires helped to craft the intervention programme that focused on exploring the teaching methods, including the 'when' and 'why' they chose those methods for various probability concepts. Data were further analyzed as to what type of manipulatives, how, why, and when they were used by these teachers. In their lessons, I focused on visual probability teaching strategies used to enhance mathematical thinking. The data gathered through lesson observations provided answers to the second research question (see Section 4.1).

The lessons observed were analyzed according to the themes that emerged from the literature review and were collated in the analytical tools (see Chapter 3) I used in this study. Each theme is presented separately by giving data in the form of excerpts taken from observed lessons. Analysis and interpretation of data ensue in each given excerpt. Data were analyzed using the two analytical tools (see Chapter 3). The description presented is structured horizontally as follows.

The excerpts of the teachers shown in the tables below gave a view of how the teachers presented their lessons, and further validated the evidence of visualisation and mathematical thinking in the probability lessons. In this section, I discussed the insights gleaned from the excerpts across the lessons as per the teaching strategy/mathematical thinking indicator observed. The horizontal analysis used to analyze the data helped to look for commonalities and differences across the lessons. It also makes it easier for a reader to understand how the participants used the visual tools to teach probability. As the analysis unfolded, different indicators of visual teaching strategies and mathematical thinking were evident.

4.3.1 Visual Probability Teaching strategies

In this study, the analytical tool (see Appendix H) developed from Batanero, Chernoff, Engel, Lee and Sánchez (2016) probabilistic reasoning and Paivio (2006). Dual Coding Theory was used to analyze data. The major themes that will be discussed in this section are: Strategies related to random events in nature, technology, and society (IRE); Teachers' use of appropriate modelling assumptions (ACR); Teachers' construction of mathematical models for stochastic situations (CMM); Application of mathematical methods and procedures of probability (AMM); Teachers' verbal and nonverbal strategies (VNVC); and an emergent theme: External visual representation (EVP).

Conversely, while I am going to discuss these themes as separate subtopics, they overlap and they share a lot in common as seen in the sections below. Furthermore, the first two probability teaching strategies given in the next two sections (see 4.3.1.1 and 4.3.1.2) are interrelated, therefore, to avoid repetition I will analyze the random events with their modelling assumptions according to their types and discuss them in Section 4.3.1.2 below.

4.3.1.1 Strategies related to random events in nature, technology, and society (IRE)

In this section, data in form of excerpts that relate to this sub-theme were identified and marked accordingly per lesson. All random events observed in all lessons were identified in this section. Codes were used to identify the relevant data, for example in Table 4.14, ***AL1-2, L1*** means Mr. Assery in lines one to two, lesson one. Table 4.14 below shows how strategies concerning IRE were used in probability lessons.

Table 4.13: Excerpts of lessons relating to random events in nature, technology, and society (IRE)

	Mr Assery	Ms Emanu	Ms Silvy
LESSON 1	▪ Reproduction in humans: XY and YY chromosomes **AL1-2, L1** ▪ Class population: 11 girls and 5 boys. **AL3, L1** ▪ Throwing a dice **AL4, L1** ▪ 6 boys sharing 2 pieces of meat. **AL6-9, L1** ▪ Tossing a coin **AL5, L1**	▪ A school selling a hundred tickets, a lucky one will win a ticket to Etosha national park for a weekend. **EL10-14, L1** ▪ Tossing a coin **EL15-17, L1** ▪ Rolling a dice **EL18-20, L1** ▪ Finding the probabilities cubes from a bag of 2 white, 2 pinks, 3 blues, and 3 purple cubes **EL21-26, L1**	▪ The probability that there will be Christmas this year. **SL29, L1** ▪ The probability that it will rain today. **SL28, L1** ▪ Probability of waking up and you are an albino. **SL27, L1** ▪ The probability that all grade 11A will pass the external examinations in 2021. **SL30, L1** ▪ The probability that grades 11A learners will not write the external examination in 2021. **SL31, L1** ▪ Probability of having snow in our town today. ▪ A box containing 4 red balls, 5 white, 2 green. **SL33, L1** ▪ Probability of choosing a Mercedes, given 5 GTI, 3 Mercedes, and 2 Jeeps. **SL34-35, L1**

	▪ Mr Assery	Ms Emanu	Ms Silvy
LESSON 2	▪ Given a universal set of natural numbers, N ≤ 10. A subset M with multiples of 3 less than 10, and a subset O with odd numbers less than 10. **AL36 - 37, L2** ▪ X is an integer. ξ = {x: 1≤ x ≤ 10} **AL38, L2** ▪ A= {x: x is a factor of 12} **AL38, L2** ▪ B= {x: x is an odd number} **AL39, L2** ▪ C = {x: x is a prime number} **AL40, L2**	▪ Tossing two dice **EL41, L2** ▪ Deck of cards EL43 - 44, L2	▪ Tossing two dice **SL45-49, L2** ▪ Bag 1 has a red, blue, and green ball each. Bag 2 has a red, blue, green, and yellow ball each. **SL50-54, L2**
LESSON 3	▪ Probability of tossing a coin twice. **AL56, L3** ▪ The probability of a test result is 0.2. However,15% of **those** with a positive result do not have the virus. 15% of those with a negative result do not have the virus. (This means that people are given incorrect results) **AL57-60, L3**	▪ Let us join groups of three so that we play a game with dice. Group one scores a point when a sum of 2-5 is rolled, group two scores a point when a sum of 6 – 8 is rolled and group 3 scores a point when a sum of 9- 12 is rolled. **EL61-65, L3**	▪ Okay, now listen to this scenario as follows: Do you know what is a kite? Okay, take note that a kite needs wind for it to fly. John enjoys flying a kite. On a given day, the probability that there is a good wind is $\frac{3}{4}$. If there is a good wind, the probability that it will fly is $\frac{5}{8}$. If there is not a good wind, the probability that the kite will fly is $\frac{1}{16}$ If the kite fly, the probability that it will stick in a tree is $\frac{1}{2}$ **SL67-73, L3**

In Table 4.13 I present data that shows the random events that I identified relating to nature, technology, and society from the three lessons for each teacher who participated in this study. I observed how the teachers used several random events to explain the topic of probability to the learners. Some of these random events were used for introducing the lessons while others were used for lesson development. More random events were employed during the teachers' first lesson when they were introducing probability to their learners. The teachers used the random events that were familiar to the learners, readily available, and those that are easily understood by the learners. In support of this, during the stimulus recall interviews, Ms Emanu revealed that "**…any readily available materials that are familiar to my learners, something that can be understood easily can be used for introducing probability"**. Similarly, Ms Silvy also acknowledged that "**…** ***I used any readily available materials that are familiar to my learners, those that can be understood easily, the ones that the learners are already exposed to"*****.**

Several random events were identified, which including scenarios, games, and practical activities. I noticed that teachers used varied random events to teach probability concepts such as single events, combined events, and probability in practice. It was evident (see Table 4.13) that learners were exposed to various random processes and interpretations to expand their understanding and enhance their thinking. Coupled with this, in most lessons I also observed that visual tools being used to aid teaching and facilitate learning. The inclusion of visuals in the lessons seemed to reinforce understanding in learners.

To avoid repetition in the next section I have analysed the random events identified in Table 4.13 as per their type and their modelling assumptions and discuss them as mentioned earlier in Section 4.3.2.

4.3.1.2 Teachers' use of appropriate modelling assumptions (ACR)

Table 4.14 below, shows how I analysed the random events according to their type (that is, the random events in nature, technology, and society). In this study random events in nature were classified by looking at aspects of nature and anything that has to do with nature, for example, genetics, rain, snow, etc. Similarly, random events in technology looked at anything that has to do with science, for example, computers, projectors, videos, and so on while random events in society looked at any situation that happens in our daily lives in our communities and things that we are mostly exposed to.

Table 4.14: Excerpts of lessons relating to appropriate modelling assumptions (ACR)

<table>
<tr><th></th><th>Mr Assery</th><th>Ms Emanu</th><th>Ms Silvy</th></tr>
<tr><td>LESSON 1</td>
<td>Random events in nature:
Reproduction in humans: XY and YY chromosomes.AL1-2, L1
Modelling assumptions:
Single events possible outcome
Chances / possibilities
Connecting probability to a real-life situation.
Probability scale from 0 to 1
Random events in society:
Class population: 11 girls and 5 boys. AL3, L1
Throwing a dice AL4, L1
6 boys sharing 2 pieces of meat. AL6-9, L1
Tossing a coin AL5, L1
Modelling assumptions:
Single events Possible outcome
Teaching from known to unknown
Probability in our daily lives (chances)</td>
<td>Random events in Society:
A school selling a hundred tickets, a lucky one will win a ticket to Etosha national park for a weekend. EL10-14, L1
Tossing a coin. EL15-17, L1
Rolling a dice EL18-20, L1
Finding the probabilities cubes from a bag of 2 white, 2 pinks, 3 blues, and 3 purple cubes. EL21-26, L1
Modelling assumptions:
Single events possible outcome/chances
Probability in real-life situations
Probability scale from 0 to 1</td>
<td>Random events in nature:
What is the probability that it will rain today? SL28, L1
What is the probability of waking up and you are an albino? SL 27, L1
What is the probability of having snow in our town today? SL31, L1
Modelling assumptions:
Understanding probability in daily lives (chances).
Random events in society:
What is the probability that there will be Christmas this year? SL29, L1
What is the probability that all grade 11A will pass the external examinations in 2021? SL30, L1
What is the probability that grade 11A learners will not write the external examination in 2021? SL31, L1
A box containing 4 red balls, 5 white, 2 green. SL33, L1
Probability of choosing a Mercedes, given 5 GTI, 3 Mercedes, and 2 Jeeps. Probability of having snow in our town today. SL34, L1</td></tr>
</table>

			Modelling assumptions: ○ Single events possible outcomes ○ Emphasize the issue of chances in our daily lives.
LESSON 2	Random events in society: Given a universal set of natural numbers, N ≤ 10. A subset M with multiples of 3 less than 10, and a subset O with odd numbers less than 10. **AL34, L1** X is an integer. ξ = {x: 1≤ x ≤ 10} **AL28, L2** ▪ A= {x: x is a factor of 12} **AL28, L2** ▪ B= {x: x is an odd number} **A39, L2** ▪ C ={x: x is a prime number} **AL40, L2** Modelling assumptions: ○ outcomes of combined events ○ Relating probability to other mathematical topics, e.g. sets, and numbers. ○ Incorporate symbols and pictures to promote mental imagery in the teaching of probability.	Random events in society: ▪ Tossing two dice. **EL41-42, L2** ▪ Deck of cards **EL43-44, L2** Modelling assumptions: ○ Possible outcomes of combined events (in abstract level). ○ Expose learners to more similar random events to expand their thinking.	Random events in society: ▪ Tossing two dice. **SL45-49, L2** ▪ Bag 1 has a red, blue, and green ball each. Bag 2 has a red, blue, green, and yellow ball each. **SL50-54, L2** Modelling assumptions: ○ Possible outcomes of combined events in (abstract level). ○ Expose learners to more similar random events to expand learners' thinking.
LESSON 3	Random events in society: ▪ Probability of tossing a coin twice. **AL56, L3** ▪ The probability of a test result is 0.2. However, 15% of those with a positive result do not have the virus. 15% of those with a negative result do not have the virus. (This means that people are given incorrect results) **AL57-60, L3** Modelling assumptions:	Random events in society: ▪ Let us join groups of three so that we play a game with dice. Group one scores a point when a sum of 2-5 is rolled, group two scores a point when a sum of 6 – 8 is rolled and group 3 scores a point when a sum of 9- 12 is rolled. **EL61-65, L3** Modelling assumptions:	Random events in society: ▪ Do you know what is a kite? Okay, take note that a kite needs wind for it to fly. Okay, now listen to this scenario as follows: John enjoys flying a kite. On a given day, the probability that there is a good wind is $\frac{3}{4}$. If there is a good wind, the probability that it will fly is $\frac{5}{8}$. If there is not a good wind, the probability that the kite

	○ Probability in practice ○ Possible chances ○ Enhancing learners' thinking through daily lives random events.	○ Probability in practice ○ Create fun and allow learners to practice. ○ Reducing probability anxiety ○ Deepens learners' probability understanding. 	will fly is $\frac{1}{16}$ If the kite fly, the probability that it will stick in a tree is $\frac{1}{2}$ **SL67-73, L3** Modelling assumptions: ○ Probability in practice ○ Connecting Probability to real-life situations. ○ Enhance learners' understanding.

Random events in nature in this study included scenarios that were employed by the teachers, for example, reproduction in humans (x and y, and y and y chromosomes and probability of being an albino) to teach the total possible outcomes for single events Society random scenarios included six boys sharing two pieces of meat, John flying a kite, selling tickets to win a ticket for Etosha National Park and a COVID-19 test in the Erongo region. Throwing a dice, tossing a coin and using coloured balls and cubes (see Table 4,14) were also analysed as society random events. There were no technology random events observed.

Regardless of the types of random events, teachers introduced and demonstrated probability concepts by emphasizing total possible outcomes for various events. This was done in a way that reinforced the understanding of the notion of chance. Total possible outcomes were then explained, using simple objects like coins, dice and coloured balls and cubes to explain less complicated scenarios like tossing a coin twice, throwing two dice simultaneously, the probability of picking balls with the same colours from two bags of coloured balls; to complicated scenarios like the COVID-19 test, six boys sharing two pieces of meat, John flying a kite, and various games (see Table 4.14).

During the teachers' first lessons, they all introduced probability by calculating possible outcomes for single events using scenarios, coins, cubes, and dice. Mr Assery and Ms Emanu used the coins and dice to calculate the possible outcomes for single events, to reinforce understanding in learners as well as enhance their thinking from two possible outcomes of a coin toss to the six total possible outcomes of a dice throw. Ms Emanu further emphasized the notion of possible outcomes by using a bag of coloured cubes to further explain the chances of picking any ball from the bag. Learners were fully engaged as they were tossing and throwing coins in the air to check the possible chances while the teacher emphasized the use of the probability formula. For dice, learners were trying to get various probabilities such as getting a five, a seven, even numbers, even numbers when you throw a dice 100 times, and many more.

Moreover, the teachers exposed the learners to questions that could help them to think more deeply to search for solutions. This created a friendly environment where learners were excited and felt at ease. The teachers emphasized the notion of chance with learners by exposing them to various examples that required both the use of visuals and scenarios to answer the questions These types of examples were arranged in such a way that they started with simple ones like the total outcomes of throwing a coin or dice, and shifted to cubes and scenarios of six boys sharing two pieces of meat, to mention a few. This was so as to allow the learners to understand the notion of chance from a concrete to a complex level and to acquire skills that could help them to deal with other mathematical topics. Across all the lessons, learners calculated various possible outcomes and presented them on the probability scale. This helped learners to give their answers in the simplest form and decimal fractions to enable them to present their answers on the probability scale. Learners were then challenged to think of any random example that they could give to the class for the fellow learners to answer. This discussion helped the learners to expand their thinking.

Conversely, society random event scenarios were used to teach probability from the social point of view which learners are mostly exposed to. With regard to Mr Assery's third example when he was presenting the probability of single events, he started as follows: ***"****Let me say you are 6 boys at home and you are given two pieces of meat, the rule may not say you should divide it equally, what is the probability that you may get a piece of meat? The soup is guaranteed; it is enough for everybody, (AL6-9, L1).*

In most cases, learners are exposed to this type of situation in their daily lives as they share food with their siblings. Since there are more boys than pieces of meat of meat, the learners were provoked to think deeply and predict the chances of getting a piece. With the help of the learners, Mr Assery facilitated the lesson to help the learners to get the answer, which they got as two over six, simplifying to 0.3 and presented it on the probability scale. This may further mean that the chances of getting a piece of meat are very minimal because 0.3 is very far from one on the probability scale. He further alerted the learners that if six boys were given six pieces to share, then they could be 100% sure that they could get a piece. By so doing the teacher enhanced the learner's thinking and deepened their understanding as they visualised their answers on the probability scale.

Similarly, in the case of a school selling tickets (see Table 4.14, Ms Emanu) made me realize the impact that probability has upon our lives. The lesson went as follows:

> Ms Emanu: *Suppose our school is selling 100 tickets, these tickets have to undergo a draw. The lucky one will win a prize ticket to spend a weekend at Etosha national park. What are your chances of getting a prize if you buy all 100 tickets?"*
>
> Learners: *100%*.
>
> Ms Emanu: *Okay, what will be your chance of getting a prize if you did not buy any ticket?*
>
> Learners: *Zero, nothing.*
>
> Ms Emanu: *Nothing neh, because you did not buy anything. If you buy twenty tickets, what are your chances of getting that prize?*

Learners: twenty *out of one hundred*.

Ms Emanu: *Yes, you are right because we have one hundred tickets.* **(EL10-14, L1)**

The learners were aware that if one buys all the tickets their chances of winning a ticket from a draw are 100% certain, similarly if one did not buy any tickets then one's chances of winning a ticket are impossible (zero).

This made it easier for the teacher to explain probability concepts faster and more easily. The case of buying twenty tickets made the teacher present the answers on the probability scale. She further explained that twenty out of a hundred has to be calculated as a fraction in its simplest form and later converted to decimal fraction before being presented on the probability scale. In that case, it simplified to 0.2 which further means that it shows minimal chances of winning a ticket because it is far from a one on the probability scale. This scenario seemed to create a positive atmosphere because it attracted the learners' attention and focus while they anticipated more to the story. Scenarios may reinforce understanding in learners. In addition, I further noticed that the teachers who participated in this study taught from what learners already knew and experienced in their communities or from what they had been taught in other subjects (like biology in this case) or other topics (like numbers and sets). By so doing, teachers helped the learners to connect probability concepts to their prior knowledge.

In another lesson, Ms Silvy introduced probability by giving its definition as a measure of how likely an event is to occur, and notably, that probability can be any number from 0 to 1 inclusively. She also introduced the probability formula to be used, for example, for any event M, P (not M) = 1 – P(M). She then asked the following questions:

What is the probability that the rain will rain today? What is the probability of you waking up tomorrow and you are an albino? What is the probability of having snow in our town today? From a box containing four red balls, five white, two green. What is the probability of picking a red ball? What is the probability of not picking red?

She then asked the learners to give any example of probability. The learners posed the following questions to the class.

What is the probability that there will be Christmas this year? What is the probability that all Grade 11A will pass the external examinations in 2021? What is the probability that grade 11A learners will not write the external examination in 2021? What is the probability of choosing a Mercedes, given 5 GTI, 3 Mercedes, and 2 Jeeps? SL27-34, L1

These questions revolved around the learners' daily experiences and prompted more learners to participate as Ms Silvy engaged them fully in the discussions. I noticed that Ms Silvy kept on pressing learners to give other probability examples and create questions around them. These questions were posed to the class and their fellow learners had to answer. For the learners, this was a fun exercise and it increased their interest because they were sharing examples of their choice. Those who could not give correct or proper probability examples were asked to try again. This helped the teacher to assess the level of understanding of the concepts in learners and to determine if she should add more examples to reinforce understanding or whether she should progress to the next challenging exercise. I think this was the best way to have all learners involved in the discussion, however, Ms Silvy's learners got too excited shouting their answers, resulting in a noisy class. She quickly noticed this and shifted to another example, to help her to regain full control of her class. This made me think that for this type of exercise a teacher needs to keep control by giving learners equal opportunities, one at a time, to avoid disruptions or spoiling the lesson.

In the same vein, I also observed that just after Ms Silvy introduced the probability formula to the learners she kept on emphasizing it to help the learners to remember the formula and to reinforce their understanding. This was evident when she asked the following questions:

> *How many balls do we have? Now, what are we favoring? What do we want to get?* She further emphasized that each one of those four balls has a chance of being picked. **SL33, L1**

This may have helped her learners to calm down and realize that probability is not something new or attached to classroom practice, but it is present in our daily lives. The teacher tried to ask questions that provoked them to think more deeply about the concept. By doing this, she fostered their learning.

It was also interesting to observe Ms Silvy emphasizing to the learners that the probability of rain depends on how often it rains in that area and whether rain is forecast. That means the probability would be fifty-fifty (it is an even chance). Also, the chance of waking up as an albino are impossible if and only if you are not an albino. Therefore, the use of multiple examples may enhance the learners' thinking and reasoning.

Moreover, their second lesson focused on the teaching of the probability of simple combined events. This time around, the teachers shifted their learners' thinking from the total possible outcomes for a single event to possible outcomes for the combined event, to promote mathematical thinking in learners. Interestingly, during the second lesson Ms Emanu and Ms Silvy used again two dice to teach simple combined events. They both rolled the two dice simultaneously and asked the learners to find the total possible outcomes for the two dice. It was also interesting to observe Ms Emanu using a deck of cards differently from the common way of finding the probability of picking a queen, a diamond, etc. She distributed four cards from each suit to thirteen learners. This helped the learners to imprint a permanent picture in their minds that 13 x 4 = 52 which is the total possible outcome for the deck of cards, and also to remember that each suit has thirteen cards. This information may be useful to any learner who is being assessed and challenged with questions of that nature. I eventually realized that learners are helped to remember the concepts taught not by using new or special teaching materials but how creatively visuals are used to enhance their thinking.

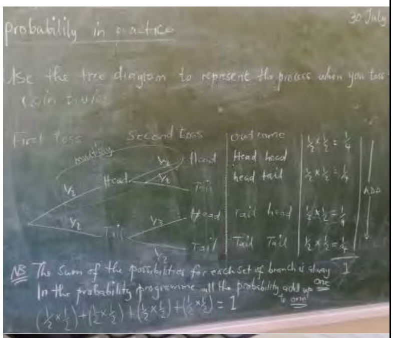
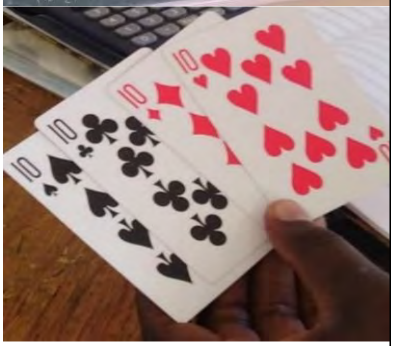
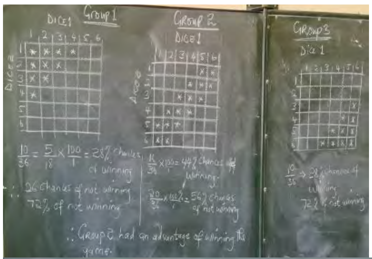
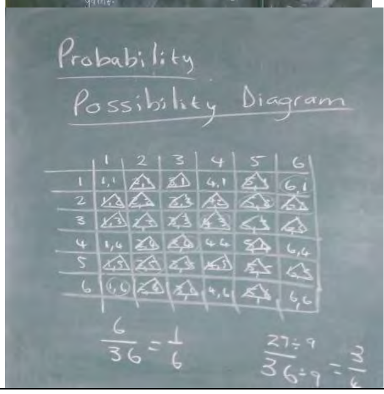
During lesson 3, the teachers focused on teaching probability in practice. They aimed to explore and develop mathematical thinking in learners by tying various probability concepts together. Learners were exposed to Tree diagrams, games and possibility diagrams to facilitate their learning as shown in Section 4.3.2.3 below.
In the next section, I present the mathematical models for stochastic situations as shown in figure 4.3.

4.3.1.3 Teachers' construction of mathematical models for stochastic situations (CMM)

Table .15 below shows how the teachers contributed to the mathematical models for random situations. It is evident from Table 4.15 that both Mr Assery and Ms Emanu used coins and dice to introduce single events. They tossed and rolled the dice to show the total possible outcomes for different tools. The teachers also explained how to find the probability of even, odd and prime numbers, and the probability of getting an even number after throwing a dice one hundred times. This was emphasized to enhance understanding and expose learners to different learning situations of. During their second lesson, both Ms Emanu and Ms Silvy used dice to teach the probability of simple combined events.

Table 4.15 Excerpts of lesson relating to construction of mathematical models for stochastic situations (CMM)

Mr Assery	**Ms Emanu**	**Ms Silvy**
▪ Throwing a dice **AL4, L1** ▪ Tossing a coin **AL5, L1**	▪ Tossing a coin. **EL15-17, L1** ▪ Rolling a dice **EL18-20, L1**	▪ Tossing two dice. **SL45-49, L2** ▪ Bag 1 has a red, blue, and green ball each. Bag 2 has a red, blue, green, and yellow ball each. **SL50-54, L2**

Mr Assery	Ms Emanu	Ms Silvy
Random events in society: ▪ Probability of tossing a coin twice. **AL56, L3**  	Random events in society: ▪ Tossing two dice. ***EL41-42, L2*** ▪ Deck of cards ***EL43-44, L2***  	 

Interestingly, I came across two similar instances from both Ms Emanu and Ms Silvy. Both teachers threw dice simultaneously, once to teach simple combined events (see Figure 4.3 and Figure 4.4 below) and drew the possibility diagrams to show the total possible outcomes of the two dice throws. Even though both Ms Emanu and Ms Silvy used the possibility diagrams to teach combined events, I noticed a difference in how they presented their possibility diagrams. I observed that Mrs. Emanu drew and labeled her possibility diagram in the presence of her learners. Columns and rows were labeled with numbers from one to six, and the teacher explained that the numbers represent the six faces of the two dice.

Furthermore, Ms Emanu further indicated all the outcomes on the possibility diagram like (6;1), (5;2), (4;3), (1;1), and so on (see Figure 4.3) which made it easier for the learners to find the probability of two numbers whose sum is seven. Because there was a visual model on the chalkboard, learners could visualise and quickly add the two numbers to get a sum of seven while the teacher circled them. Similarly, the learners could judge and pointed out the wrong answers when the teacher circled a wrong answer by mistake:

> Learner: *it is forming a diagonal Ms!*
>
> Ms Emanu: *you are right.* **EL41-42, L2**

The circles for the sum of seven formed the diagonal, which the learners quickly noticed and this was because of the visual model that the learners could judge and assess the situation at hand. This may could also expand their way of reasoning which may lead to enhancing their thinking. Below is Ms Emanu's mathematical model that involved two dice for stochastic situations as shown in Figure 4.3 below.

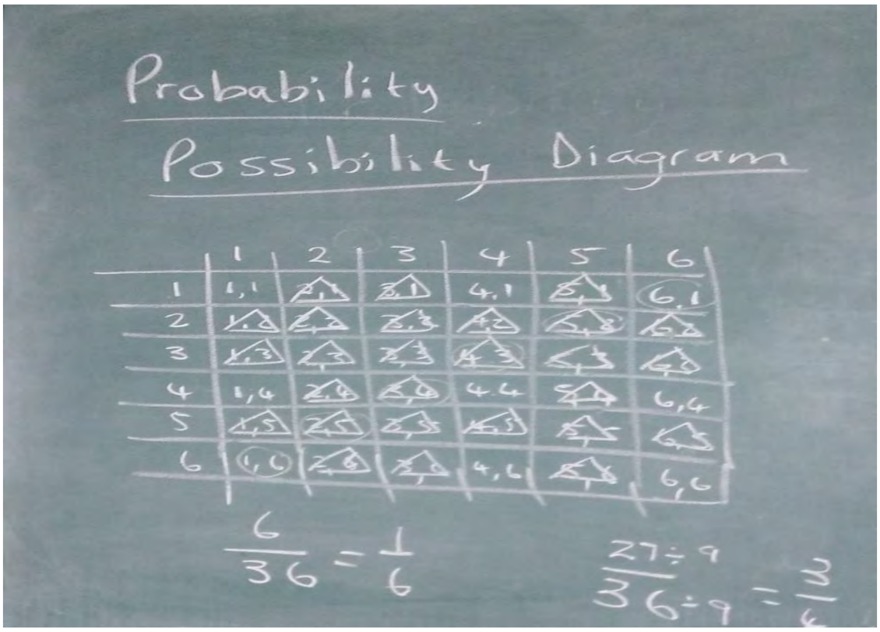


Figure 4.3: Ms Emanu's possibility diagram for the two dice thrown simultaneously.

Several questions were posed to the learners to find the probability for at least a one, double fours, the sum of seven and the sum of eight, and the patterns were observed. The answers for the probability of prime numbers were marked with triangles, to avoid confusion when the learners were identifying the primes and the digits whose sum is seven. As a result, the learners noticed another pattern when all the rows and columns for prime numbers were filled with triangles, while the non-prime number rows and columns had three triangles each. The learners were exceptionally engaged and observant trying to predict how other patterns could be formed, while expanding their thinking in the process. After observing the model, they realized that the total possible outcomes for the two dice throws are thirty-six (and not twelve as they whispered earlier).

However, I felt that the model was not explored enough because Ms Emanu could have gone further to make a comparison between finding the probability of the sum and the difference of any number in the model. Learners could have observed how the patterns of such probabilities behave, for example, every probability of the sum of five, six, seven or eight formed the anti-diagonal or counter-diagonal running from the top right to the bottom left corner of the model. Similarly, the probability for any difference also formed a main or leading diagonal as they ran from the top left to the bottom right corner of the model. Therefore, Ms Emanu could have also linked these diagonals to the positive and negative gradients of the straight-line graphs (that they had learned in graphs of functions) to ensure that the learners' mathematical thinking was fully enhanced.

Conversely, Ms Silvy 's stochastic model in Figure 4.4 below was drawn and marked with stars well in advance before the learners came into her class. This was done to avoid spending a lot of time on drawing instead of explaining and discussing the concept. A dice was thrown and learners were asked to give the total possible outcomes for throwing a dice, to which the learners responded as six.

By so doing, the teacher checked on the learners' prior knowledge with dice which could have helped to shape their ideas when thinking towards the throwing of two dice simultaneously. She then threw two dice and asked the learners to find the total possible outcomes, to which they responded as twelve (6+6). I observed that the learners could not relate the question to the possibility diagram on the chalkboard and that seemed to be the reason why they kept on guessing the answers as 12 (6 + 6), 1 (6÷ 6), and so on.

Moreover, looking at Ms Emanu and Ms Silvy's presentations (see Figures 4.3 and Figure 4.4) made me realize that mathematical models helped the learners to think more deeply, in the sense that they had to recall the topics covered in previous lessons. Figure 4.4 below shows Ms Silvy's mathematical model for stochastic situations that involved the two dice thrown simultaneously.

Figure 4.4: Mrs. Silvy's mathematical model for stochastic situations.

Furthermore, during this lesson, Ms Silvy made several mistakes when she was circling the answers on the model, but because the model was visual and on the chalkboard, learners could judge and rectify the teacher as is evident from these excerpts:

> Learner: *Ms, it is going in the diagonal!*
>
> Ms Silvy: *Thank you for noticing that.*
>
> Learner: *Ms Are we just going to continue with the old formula?* ***SL45-49, L2***

I observed learning was taking place as learners reasoned to motivate their answers, and as a result, they identified that some of the answers were omitted, resulting in incorrect calculations. The learners further suggested that the teacher to start again and cancel out all the rows and columns for prime numbers to find the P (at least odd). While looking for the P (at least prime) learners noticed that (6,2); (6,4); (6,6); (4,4) (4,6); (2,4); (2,6); (4,2) and (2,2) were not part of the required set, so they told the teacher to cancel them out.

> Learner: *Ms not everything there, do not cancel everything in that column!* **SL45-49, L2**

From my point of view, these types of models have diverse possibilities and many questions can be answered through using such models. I noticed that there were some missed opportunities by these two teachers. The study wanted them to see how they could use these models to enhance mathematical thinking but it seemed to me that some teachers did not do this. Hence, I felt Ms Silvy should have taken the ideas further than what she presented. Additionally, while one of her learners identified that the P (sum of 7) formed a diagonal, I felt that was the best opportunity to expose her learners to different probabilities, like finding the P (two digits whose product is a multiple of two) or P (of two digits divisible by three), or they could even generate a formula from those patterns so that learners could identify more patterns to explore the beauty of mathematics and learn in the process.

Both Ms Emanu and Ms Silvy gave had their second mathematic model for the stochastic lesson to teach simple combined events using different visual tools to present their lessons (see Figures 4.5 and Figure 4.6). This time around Ms Emanu used the deck of fifty-two cards (see Figure 4.5). She distributed the cards among thirteen learners which were each given a different suit with thirteen cards each. Ms Emanu used this example to emphasize the concept of the total possible outcomes and at the same time exposed learners to the deck of cards to enable them to answer the questions related to the cards. Figure 4.5 below shows Ms Emanu's second mathematical model for stochastic situations that involved cards.

Figure 4.5: Ms Emanu's second mathematical model.

This experience could also help learners to visualise and deepen their understanding of chance at an abstract level. Cards could further help the learners to relate their learning to the games with cards that they play outside the classroom environment, and apply their skills to answer the questions. Similarly, cards may also help the learners to expand their thinking levels as they visualise the cards and respond to questions like: what is the probability of a red card or a king? During this lesson, I was extremely keen to see how Ms Emanu used cards to explain the P (red card or a king). After Ms Emanu's presentation, I felt that she should have continued extending learning by presenting her answers on the Venn diagrams to show the overlaps of red cards and a king. This could have been illustrated by showing the two sets intersecting each other, whereby two sets represent the red cards and kings respectively. The intersection would then be the double count for the red king. Ms Emanu would then have shaded the two sets that would result in the intersection being shaded twice, and then explained the aspect of $\frac{30}{52}$ - $\frac{2}{52}$ =$\frac{7}{13}$. By so doing, she would have enhanced the learners' mathematical thinking.

In addition, Ms Silvy further used the two bags, with one bag containing red, blue, and green balls while the other bag contained red, green, blue, and yellow balls (see Figure 4.6 below). She tried to reinforce understanding in learners by giving them multiple examples, yet exposing them to different ways of thinking. The teacher's interaction with the learners through discussions thus helped the learners to gain confidence and participate in the lesson. This created a friendly and conducive environment for learning. I observed that learners were participating freely by suggesting answers to the teacher. However, I also noticed that some learners were also admiring the balls, therefore there may be a need to be extra careful when using these tools to avoid distractions in learners. Practical examples may also help the teachers to engage learners in realistic situations and with the facilitation of lessons. This was evident as the learners responded faster to the questions. This is shown in Figure 4.6 below as Ms Silvy was showing the balls to the learners to answer the questions, using her second mathematical model for stochastic situations.

Figure 4.6: Ms Silvy's possibility diagram for the coloured balls.

Similarly, for her next model with two each of red, blue, green, and yellow balls, Ms Silvy wanted to emphasize the aspect of 'at least', so she calculated the probability of one red ball and left out the other one on purpose. The learners saw that the red ball was being left out and so they quickly pointed it out. They were focused and this was evident from what they said:

Learner: *But there is another red Ms?*

Ms Silvy: *Thank you! You have to be vigilant guys.*

Learners were further asked to give the P (same colour). They could see the colours on the model, so they pointed out all the possibilities as follows.

Learners: *Ms It will be (blue, blue), Ms, the red also (red, red), (green, green).*

Because they could see it on the chalkboard and the teacher had focused on the visual aspect, learners were able to assess what the teacher had done.

Ms Silvy: *How many did you count?*

Learner: *Six, Ms, those two are also included.*

Ms Silvy: *This one also? Are you sure?*

Learner: *Yes, Ms, it is six green balls out of twelve balls.*

Furthermore, after the teacher corrected her mistake on the board, a certain leaner stood up to confirm the corrected answer.

Learner: *Ms Did you also circle the yellow?*

Ms Silvy: *No, yellow is out it is not part of that.*

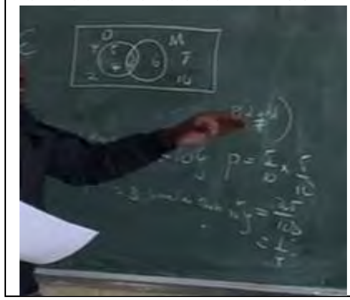
As much as Ms Silvy tried to expose learners to various activities using the models, I still think these models were not explored enough with the learners. Several questions could have been posed to utilize the models fully, to expand the learner's mathematical thinking. This could have been useful to the learners when trying to build their understanding of a particular concept. The two mathematic models used by Ms Silvy and Ms Emanu made me think that they could be also have been done by tossing two coins with learners having to visualise the possible outcomes for the coins like HH, TH, HT, TT, and so forth, Similarly, since Ms Emanu and Ms Silvy used similar models (see Figures 4.3 and Figure 4.4), the methods that I suggested earlier for the model in Figure 4.3 to expand the learners thinking further also applies to Figure 4.6.

In this teaching strategy, I focused on how the teachers contributed to the use of random event mathematical models. This included the lessons that integrated the use of visual tools for stochastic situations, for example, tossing coins, rolling dice, choosing playing cards, picking coloured balls, etc. I observed how these tools were used to facilitate learning and learners were given enough time to explore their learning. However, this was only moderately observed because few activities were given to the learners to tackle various probability tasks. The following section looks at the application of mathematical methods and procedures of probability.

4.3.1.4 Application of mathematical methods and procedures of probability (AMM)

Table 4.16 below indicates how practical work was administered to the learners in various lessons. The extracts of lessons in Table 4.16 further shows how the three teachers administered practical activities to the learners. T_{15} suggested in the questionnaire that for the teachers to improve the teaching of probability they need to use plenty of visual materials, different methodologies and connect probability to real-life situations because in most cases learners tend to ask why should they learn probability.

Table 4.16: Excerpts of lessons relating to mathematical methods and procedures of probability (AMM)

Mr Assery	Ms Emanu	Ms Silvy
Given a universal set of natural numbers, N ≤ 10. A subset M with multiples of 3 less than 10, and a subset O with odd numbers less than 10. **AL34, L1** X is an integer. ξ = {x: 1≤ x ≤ 10} **AL28, L2** ▪ A= {x: x is a factor of 12} **AL28, L2** ▪ B= {x: x is an odd} **A39, L2** ▪ C = {x: x is a prime} **AL40, L2** Mr Assery: (looking at learners' work) It is fine, why? Is it not a prime number? Aaah, Jesus, you see now? Okay, look here, use the notation to complete the statement. Group 2: Let's see, okay very technical. You see these things you start with, it is just a slight mistake, where does it belong? Group 3: Okay, you are fine, aah, come on. Yeah, this is fine, thus fine, thus beautiful, thus also okay, aah it looks nice, wuuuh, perfect, woow good. This is the only thing that was not supposed to be there, other than that, the story is nice. Wonderful. **AL36 - 40, L2** 	Let us join groups of three so that we play a game with dice. Group one scores a point when a sum of 2-5 is rolled, group two scores a point when a sum of 6 – 8 is rolled and group 3 scores a point when a sum of 9- 12 is rolled. **EL61-65, L3** Teacher: group 1 could score a point if they either got a sum of 2, 3, 4, and 5; group 2 could get a point if either they got a sum of 6, 7, and 8 while group 3 could get a point if they either got a sum of 9,10,11 and 12. By looking at the rules for the games provided, one would think group 1 and 3 had an advantage of scoring comparing to group two. Therefore, she pointed out that both group 1 and 3 had only 10 out of 36 (28%) chances of winning and 26 out of 36 (72%) chances of not winning (see Figure 4.12) while group 2 had 16 out of 36 (44%) chances of winning and 20 out of 36 (56%) chances of not winning, **EL61-66, L3**	Okay, guys, the rule is very simple, always make sure that each branch of a tree diagram adds up to 1. As you go along the branches, you should always multiply the probabilities while when you go down the branches you should always add the probabilities. Is it clear? Is it making sense? **SL67-73, L3** Learner: ohoo, so you first solve the first branch? Teacher: If you know the probability of not flying, P (not flying) then you can get the P(flying) = …? Teacher: Can you please complete another branch? Just go ahead Teacher: Can you just finish up everything? Teacher: Can you quickly explain where you are getting those answers, others also want to know. Teacher: The P(not flying)?

Mr Assery	Ms Emanu	Ms Silvy
▪ The probability of a test result is 0.2. However, 15% of those with a positive result do not have the virus. 15% of those with a negative result do not have the virus. (This means that people are given incorrect results) **AL57-60, L3**		Learner: Then you just continue again with the formula, then you say 1- $\frac{1}{16}$ = $\frac{15}{16}$ Teacher: Good girl, I am so excited! Thank you very much. **SL67-73, L3**

Mr Assery gave the learners activities in groups on the Venn diagram (see Figure 4.7 and Figure 4.8) and Tree diagrams (Figure 4.9) below. Figure 4.7 below shows how HE demonstrated and facilitated the lesson by giving examples on the chalkboard before the group work activity.

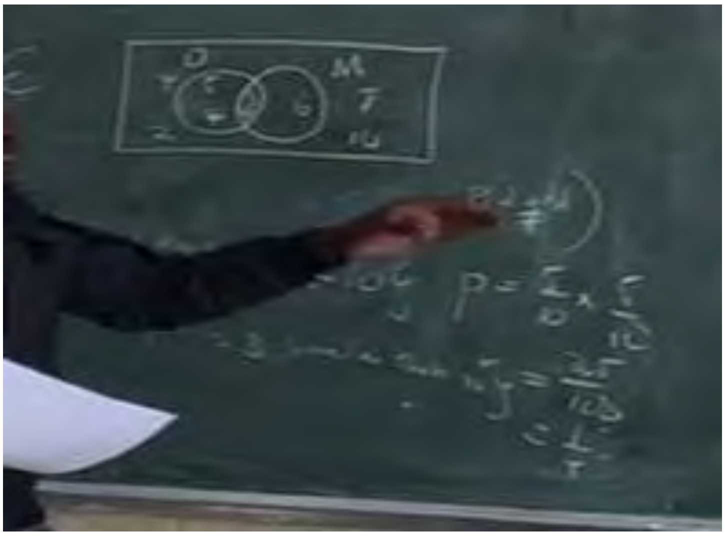


Figure 4.7: Mr Assery demonstrating combined events on the chalkboard using a Venn diagram.

The Venn diagram (see Figure 4.7 and Figure 4.8) helps learners to visualise probabilities and sort numbers into their respective sets. Learners may also be able to recall the skills they were taught in topics like numbers and sets, and apply them to help them to find answers in probability. This may help them to link the new knowledge to prior knowledge through critical thinking. Moreover, the appearance of Venn diagrams may attract the learners' attention, help them to focus, and to anticipate what is next. This could help learners to concentrate and ask questions about challenging areas. This may expand their thinking.

Additionally , learners were then given group work to test their abilities while they were encouraged to discuss actively: X is an integer. $\xi = \{x: 1 \leq x \leq 10\}$ AL28, L2; A= $\{x$: x is a factor of 12$\}$ AL28, L2; B= $\{x$: x is an odd$\}$ AL39, L2; C = $\{x$: x is a prime$\}$ **AL40, L2**. By so doing, Mr Assery facilitated the methods and procedures acquired when learners were taught sets by letting them demarcate the elements of sets given in notation form and calculate the probability thereof. These types of activities engage the learners in debates when trying to get answers – thus expanding their mathematical thinking. Figure 4.8 below shows how Mr Assery supervised the groups.

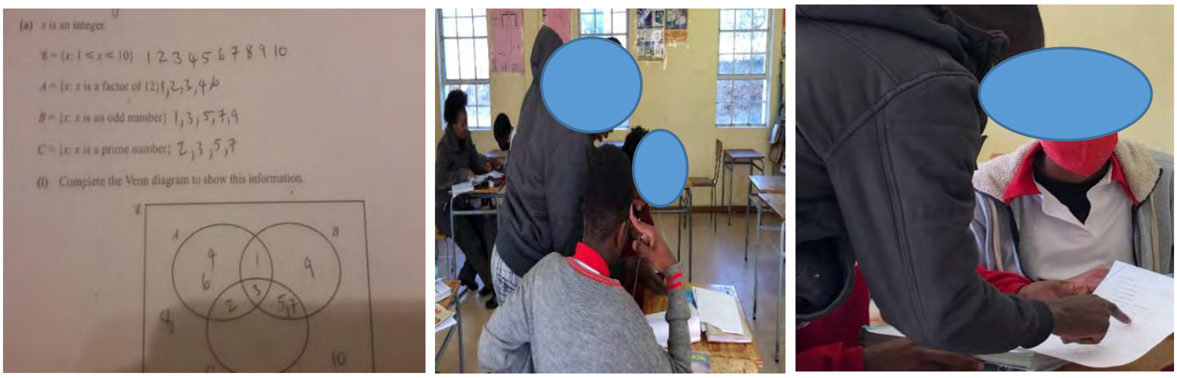


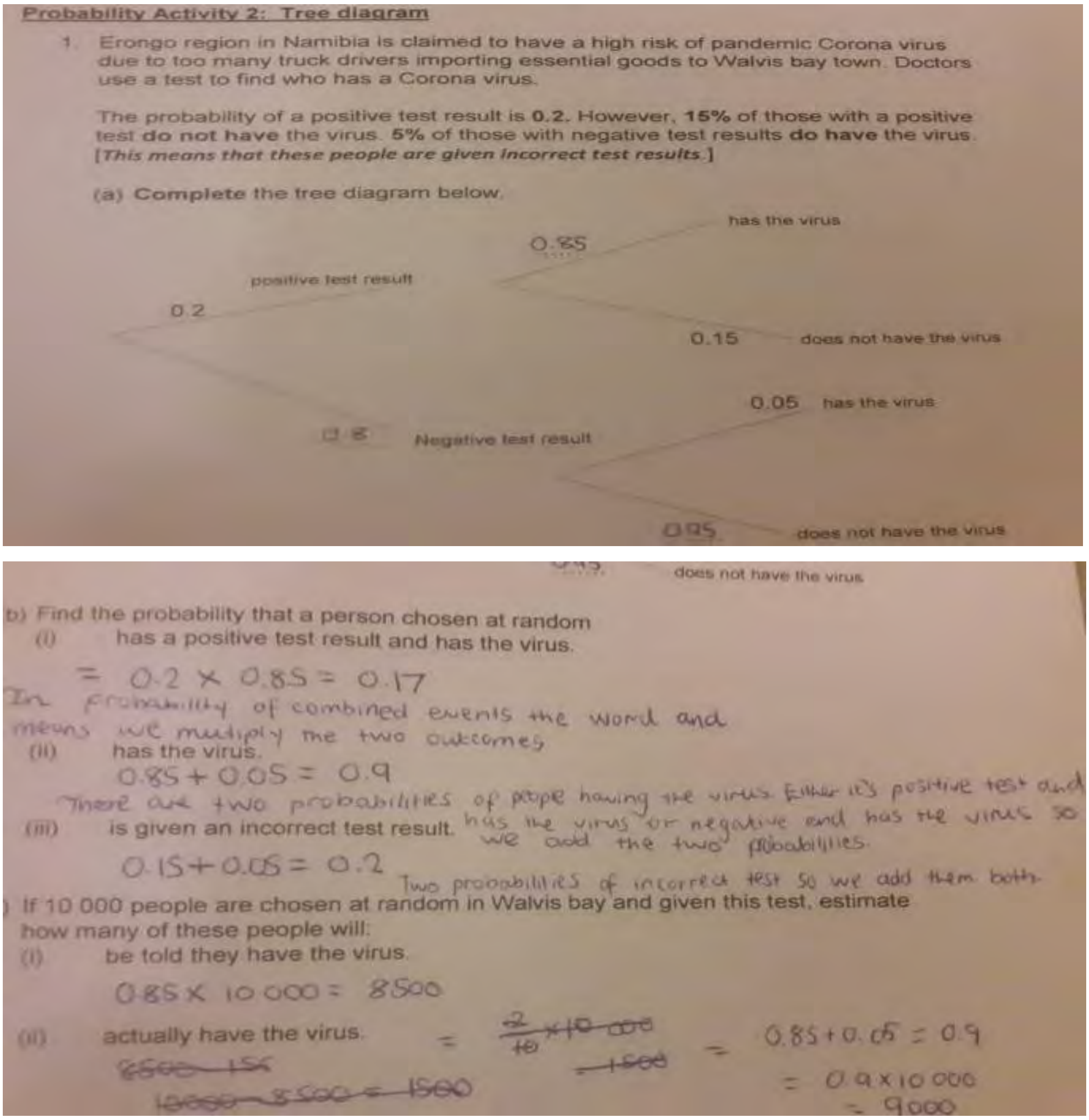
Figure 4.8: Mr Assery moving around the groups checking the learners' work.

Mr Assery also encouraged the learners and facilitated them in approaching various probability tasks when he walked around the groups. (see Figure 4.8). This was evident as I observed him commending on the learners' work as follows:

> Mr Assery: (looking at learners' work) *Group 1: It is fine, why? Is it not a prime number? Aaah, Jesus, you see now? Okay, look here, use the notation to complete the statement. Group 2: Let's see, okay very technical. You see these things you start with, it is just a slight mistake, where does it belong? Group 3: Okay, you are fine, aah, come on. Yeah, this is fine, thus fine, thus beautiful, thus also okay, aah it looks nice, wuuuh, perfect, woow good. This is the only thing that was not supposed to be there, other than that, the story is nice. Wonderful*. **AL36 - 40, L2**

Group work seemed to be a good idea because learners discussed freely with their peers, which awarded them time to explore the concept and ask others to explain their answers and reasoning.

Mr Assery further engaged the learners with Tree diagram activities which he first discussed with them. Their discussions attracted the learners' attention as they tried to relate to and understand the current COVID-19 testing statistics in the country. This also encouraged them to actively discuss, trying to think outside the box, as they responded to the given questions (see Figure 4.9). Figure 4.9 below shows the group work that Mr Assery administered to his learners.



Probability Activity 2: Tree diagram

1. Erongo region in Namibia is claimed to have a high risk of pandemic Corona virus due to too many truck drivers importing essential goods to Walvis bay town. Doctors use a test to find who has a Corona virus.

The probability of a positive test result is **0.2.** However, **15%** of those with a positive test **do not have** the virus. **5%** of those with negative test results **do have** the virus. [*This means that these people are given incorrect test results.*]

(a) Complete the tree diagram below.

b) Find the probability that a person chosen at random
(i) has a positive test result and has the virus.

= 0.2 × 0.85 = 0.17

In probability of combined events the word and means we multiply the two outcomes

(ii) has the virus.

0.85 + 0.05 = 0.9

There are two probabilities of people having the virus. Either it's positive test and has the virus or negative and has the virus so we add the two probabilities.

(iii) is given an incorrect test result.

0.15 + 0.05 = 0.2

Two probabilities of incorrect test so we add them both.

) If 10 000 people are chosen at random in Walvis bay and given this test, estimate how many of these people will:
(i) be told they have the virus.

0.85 × 10 000 = 8500

(ii) actually have the virus.

~~8500 − 15~~

~~10000 − 8500 = 1500~~

$= \cancel{\frac{2}{10} \times 10\,000}$

~~= 1500~~

= 0.85 + 0.05 = 0.9

= 0.9 × 10 000

= 9000

Figure 4.9: Mr Assery's partially completed Tree diagram that was given to the learners.

In part (a) of the activity in Figure 4.9 above, learners were presented with the partially completed Tree diagram. I can conclude that the teacher facilitated the application of methods and procedures because learners had to figure out that for them to complete the branch of the Tree diagram, they would have to use the formula P(E) = 1- P (Not E), whereby P(E) is the probability of an occurring event. Similarly, Part (b) of the above activity revealed the learners applying the probability procedures for **OR** and **AND.** Even though some answers were not calculated correctly, one could deduce that these learners were thinking mathematically because they knew the approaches/procedures to tackle this type of problem. Consequently, learning to think mathematically is not about getting answers (Devlin, 2012), though once you have learned to think mathematically, getting the right answer becomes a lot easier than when you are merely following procedural recipes. Mathematical thinking may therefore be viewed as a process of trying and reflecting but not getting answers.

I was therefore prompted to think that an investigation like a scenario of 'doctors in the Erongo region using a test to find who has a COVID-19 (see figure 4.9) may not only teach learners how to find the answers but may also contribute to the expansion of learners' knowledge and thinking. Similarly, this may also lead to learners' understanding of making predictions in the future that may influence their decision-making. Learners may be encouraged to understand and question the world in which they live – in this case they could learn that even if people are being tested positive for COVID-19, there are also chances that they might be given the wrong results – thus encouraging thinking outside the box. Having said that, I support the study done by Nicholson (2005) and Taylor (2011) that emphasized that probability plays a major role in modern society, both in the daily lives of the public at large and in professionals' activities within society. This refers to the situation explained above, which seemed to encourage teaching for understanding.

On the same note, Ms Silvy also taught probability in practice by using a Tree diagram (see Figure 4.10). I observed that although both Mr Assery and Ms Silvy used Tree diagrams to teach probability in practice, they did it in different ways. Mr Assery administered group work while Ms Silvy invited a volunteer (one of the learners) to the chalkboard to complete the Tree diagrams and explain to others how she got the answers. In so doing, Ms Silvy facilitated the probability formula that learners consistently kept on using to complete the Tree diagrams and the **OR** and **AND** rule for probability. This was evident as I observed her informing the learners as follows:

> Ms Silvy: *Okay, guys, the rule is very simple, always make sure that each branch of a tree diagram adds up to 1. As you go along the branches, you should always multiply the probabilities while when you go down the branches you should always add the probabilities. Is it clear? Is it making sense?* **SL67-73, L3**

Both methods (see Figure 4.9 and Figure 4.10) seemed to work because I observed that in both incidences learners participated well. It was interesting to observe a learner freely presenting his ideas on the chalkboard and giving opportunities to other learners to answer the questions. Ms Silvy facilitated the applications of mathematical methods and procedures of probability by asking the following questions:

> Ms Silvy: *If you know the P (not flying), then you can get the P(flying) = …?*
>
> Learner: *okaaay,*
>
> Ms Silvy: *Can you quickly explain where you are getting those answers, others also want to know.*

Learner: *Because it is for this branch you see, if you read the statement for the information you are given, you will see here no good wind, then you check here on the branch no good wind, then you put the probability that you were given.*

Ms Silvy: The P (not flying)?

Learner: *Then you just continue again with the formula, then you say 1-* $\frac{1}{16} = \frac{15}{16}$

Ms Silvy: *Good girl, I am so excited! Thank you very much.* **SL67-73**

Figure 4.10 below shows how Ms Silvy and her learner presented the lesson using a Tree diagram.

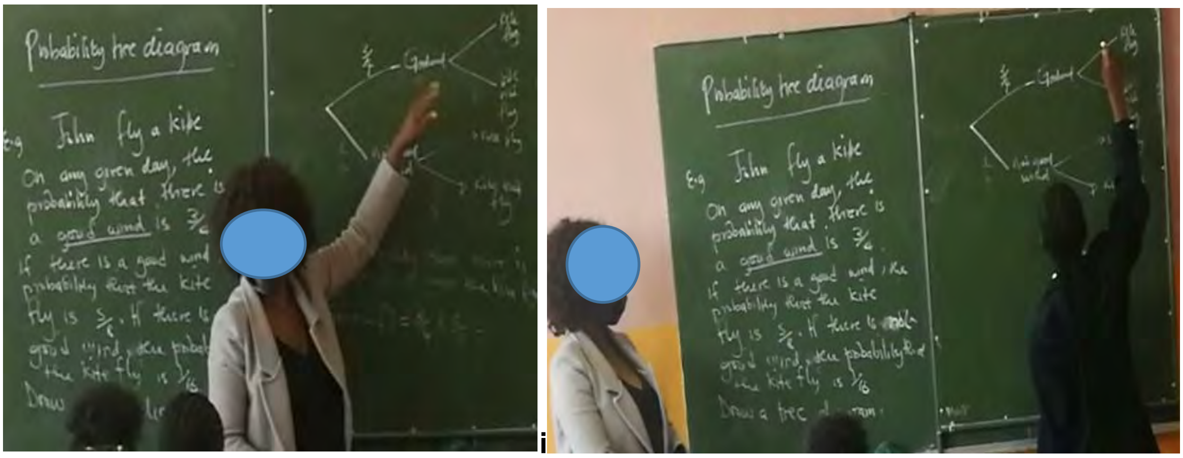


Figure 4.10: Ms Silvy and a learner completing a Tree diagram.

This type of teaching encourages learner-centered education where learners are more involved in their learning which leads to them creating their knowledge as they communicate. Learners also expand their thinking and deepen their understanding in the process as they explain their reasoning to others. I also noticed that learners were so impressed, relaxed, and happy to listen to their fellow peers, that they were encouraged to ask questions and learn from others as well. As they engaged in thinking, this activity further allowed the learners to acknowledge the concept of probability. The lesson promoted thinking as they tried to reason around the problem of the kite needing wind for it to be able to fly or fly and get stuck in the tree. This enhances their understanding.

Lastly, Ms Emanu introduced a probability game with dice to the learners and told them to answer the questions afterwards. By looking at Figure 4.11 below, it is evident that Ms Emanu used the game approach to facilitate the applications of mathematical methods and procedures of probability. Figure 4.11 below shows the learners playing a probability game in class.

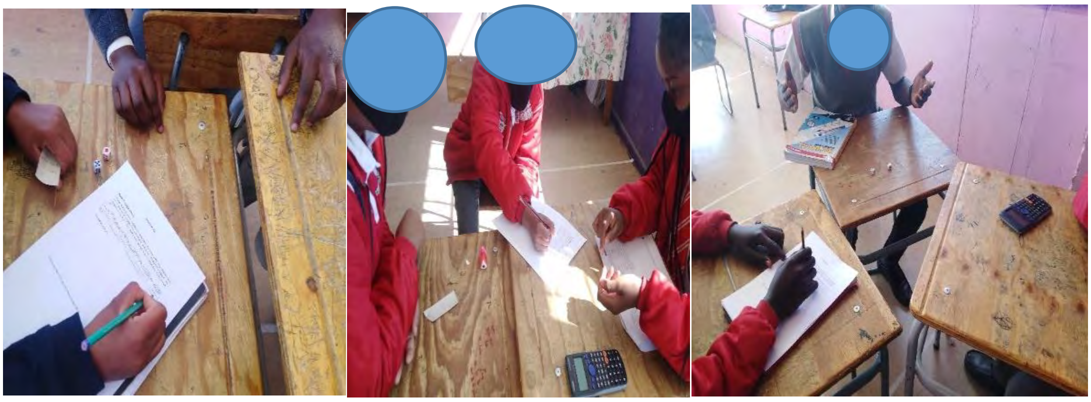


Figure 4. 11: Learners playing probability games in class

As dice are very common and learners often play games with dice at home, this forced Ms Emanu to change the rules for the game so that the learners could learn something new. To minimize the risk of the learners becoming bored by using familiar games, Ms Emanu tried to be creative with games. She thus offered small variations on standard experiments (games) and usual games to motivate her learners, thereby and increasing their curiosity, thinking and learning in the process. On the same note, employing various approaches to probability seemed to help the teachers to develop a strong, coherent, and intuitive pedagogical knowledge as well as simulation tools to enable them to teach successfully and make learners understand probability concepts (Naresh et al., 2014).

Likewise, T_{30} revealed that learners should be actively engaged in constructive learning, given enough time to think and to practice, raise questions that require critical thinking to enhance mathematical thinking, assess the learners regularly and give feedback promptly. Coupled with that, T_8 also revealed in the questionnaire that probability games help to improve the teaching of probability. Learners were then asked the following questions:

> MS Emanu: *How did the game go? Does it seem fair? If not, who has the advantage? Why?* **EL61- 65, L3.**

I observed that learners could see that the game was not fair because they were not scoring more points or they only had a few points. Mrs. Emanu further gave feedback (see Figure 4.12) to the learners so that they could understand more clearly and to dispel misconceptions. This was done to emphasize the aspect of randomness.

Ms Emanu drew the possibility diagrams (see Figure 4.12) for each group to show their chances of winning. This was done to allow learners to observe and compare the results of the three groups and think deeply about what each group's chances of winning were. In this way, the learners were appeased.

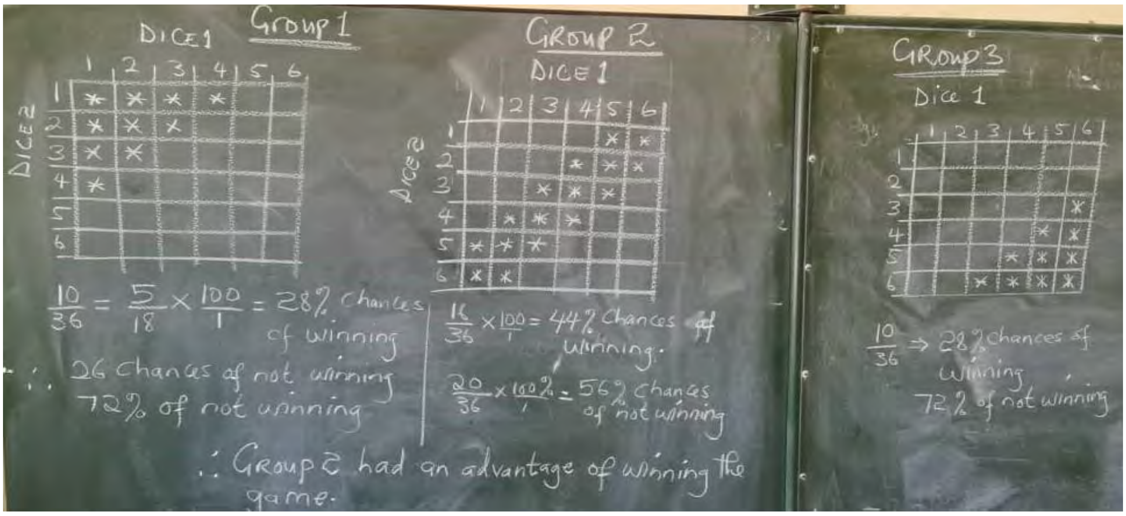


Figure 4.12: Ms Emanu showing which group had the advantage to win the game

This also made me think that Group 2 had an advantage of winning the game because it had few chances (56% compared to 72%) of losing (not winning). However, even though all three groups had seventeen rounds to throw the two dice simultaneously, simply because one group had to win does not mean Group 2 should win in that specific lesson. This is because random events need to be repeated many times to get the results one wants. Similarly, one may not get the desired results if you only examine a few experiments. This was also evident because Group 2 had sixteen chances of winning and after throwing the dice for seventeen rounds they still had several zeros recorded on their worksheet. With that in mind, one can conclude that the game was NOT fair because all groups did not have equal chances of winning and even if you have a high chance of winning, you also have to try several times before celebrating a victory. This could also help the learners to think more deeply that both groups 1 and 3 were in better positions to compete because they had equal chances of both winning and losing. Rich real-life situations and games help to facilitate learners' understanding of probability. This further supports the study done by Devlin (2011) that acknowledged that games can facilitate mastery of basic probability skills in a much more rapid way and with longer-lasting effects if that practice is done in the pursuit of a meaningful goal rather than mere repetition for its own sake.

This teaching strategy is aimed at checking whether teachers are able to administer practical investigations and projects to learners and present their work in class. During my observation, these incidences were minimal. This was because each teacher was observed giving two, or fewer than two practical activities to the learners. These included a game with dice, investigations like the COVID-19 situation, John flying a kite and working with the Venn diagram. Only one lesson was observed with a learner presenting to the whole class.

4.3.1.5 Teachers' verbal and nonverbal strategies (VNVC)

Table 4.17 below shows the verbal and nonverbal codes used by the teachers to teach probability. I noticed that some verbal and nonverbal codes were interchanged within each lesson while others were interchanged from lesson to lesson. This was evident when the teachers consistently used concrete materials like coins, dice, cubes, decks of cards, balls, etc. Similarly, teachers also used diagrams like probability scales, possibility diagrams, Tree diagrams and Venn diagrams for learners to visualise and enhance their thinking. In addition, employing verbal and nonverbal codes in teaching helps learners to understand the concept better. I therefore, indicate these codes in Table 4.17. This made me think further that the integration of visuals (concrete and diagrams) from the initial lesson planning stages to the lesson delivery or presentation may cause learners to master the content and expand their learning. Furthermore, any concrete materials and diagrams observed during the observation period fell under nonverbal codes. These included coins, dice, cards, cubes, balls, and diagrams such as probability scales, Venn diagrams, possibility diagrams and Tree diagrams.

Table 4.17 Excerpts of lessons relating to verbal and nonverbal strategies (VNVC)

	Mr Assery	Ms Emanu	Ms Silvy
LESSON 1	▪ Reproduction in humans: XY and YY chromosomes **AL1-2, L1 (verbal code)** ▪ Class population: 11 girls and 5 boys. **AL3, L1 (nonverbal code)** ▪ Throwing a dice AL4, L1 (Nonverbal code) ▪ 6 boys sharing 2 pieces of meat. **AL6-9, L1 (verbal code)** ▪ Tossing a coin AL5, L1 **(nonverbal code)**	▪ A school selling a hundred tickets, a lucky one will win a ticket to Etosha national park for a weekend. **EL10-14, L1 (verbal code)** ▪ Tossing a coin EL15-17, L1 (n**onverbal code)** ▪ Rolling a dice EL18-20, L1 (Nonverbal code) ▪ Finding the probabilities cubes from a bag of 2 white, 2 pinks, 3 blues, and 3 purple cubes **EL21-26, L1 (Nonverbal code)**	▪ The probability that there will be Christmas this year. **SL29, L1 (verbal code)** ▪ The probability that it will rain today. **SL28, L1 (verbal code)** ▪ Probability of waking up and you are an albino. **SL27, L1 (verbal code)** ▪ The probability that all grade 11A will pass the external examinations in 2021. **SL30, L1 (verbal code)** ▪ The probability that grades 11A learners will not write the external examination in 2021. **SL31, L1 (verbal code)** ▪ Probability of having snow in our town today. **(verbal code)** ▪ A box containing 4 red balls, 5 white, 2 green. **SL33, L1 (verbal code)** ▪ Probability of choosing a Mercedes, given 5 GTI, 3 Mercedes, and 2 Jeeps. **SL34-35, L1(verbal code)**
LESSON 2	▪ Given a universal set of natural numbers, N ≤ 10. A subset M with multiples of 3 less than 10, and a subset O with odd numbers less than 10. **AL36 - 37, L2 (verbal code)** ▪ X is an integer. $\xi = \{x: 1 \leq x \leq 10\}$ **AL38, L2 (verbal code)** ▪ A= {x: x is a factor of 12} **AL38, L2 (verbal code)** ▪ B= {x: x is an odd number} **AL39, L2 (verbal code)** ▪ C = {x: x is a prime number} **AL40, L2 (verbal code)**	▪ Tossing two dice EL41, L2 **(nonverbal code)** ▪ Deck of cards EL43 - 44, L2 **(nonverbal code)**	▪ Tossing two dice ***SL45-49, L2*** ▪ Bag 1 has a red, blue, and green ball each. Bag 2 has a red, blue, green, and yellow ball each. ***SL50-54, L2 (verbal code)***

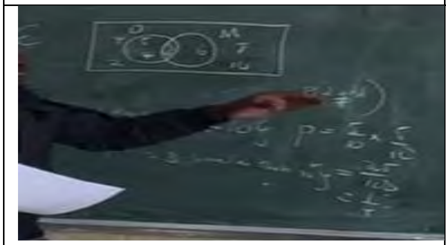
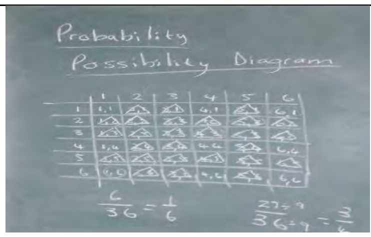
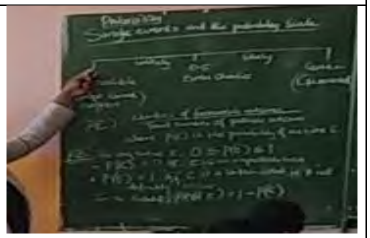
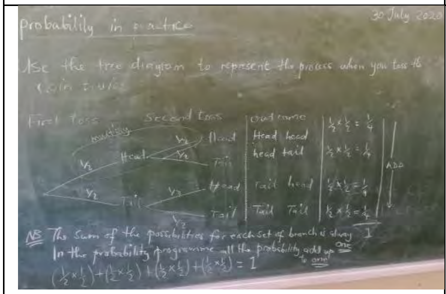
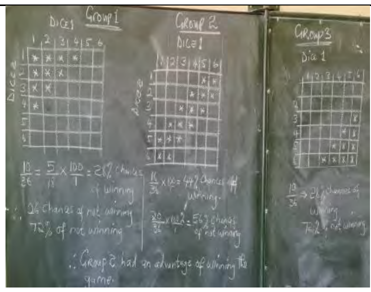
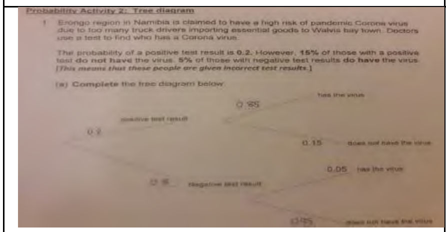
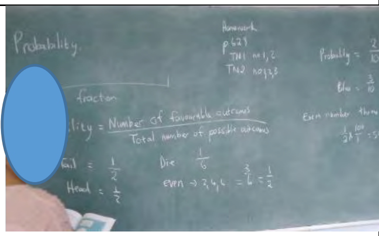
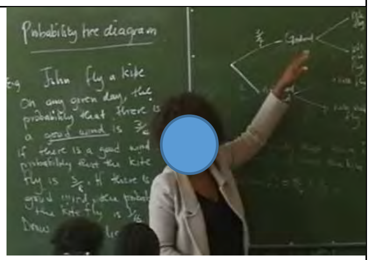
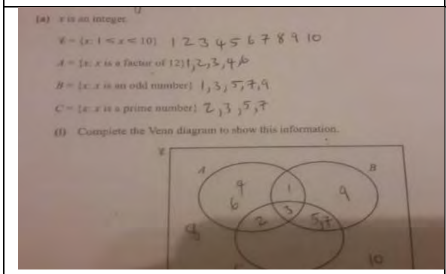
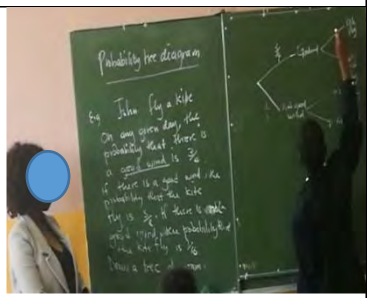
	Mr Assery	Ms Emanu	Ms Silvy
LESSON 3	▪ Probability of tossing a coin twice. **AL56, L3 (Nonverbal code)** ▪ The probability of a test result is 0.2. However,15% of those with a positive result do not have the virus. 15% of those with a negative result do not have the virus. (This means that people are given incorrect results) **AL57-60, L3 (verbal code)**	▪ Let us join groups of three so that we play a game with dice. Group one scores a point when a sum of 2-5 is rolled, group two scores a point when a sum of 6 – 8 is rolled and group 3 scores a point when a sum of 9-12 is rolled. **EL61-65, L3 (nonverbal code)**	▪ Okay, now listen to this scenario as follows: Do you know what is a kite? Okay, take note that a kite needs wind for it to fly. John enjoys flying a kite. On a given day, the probability that there is a good wind is $\frac{3}{4}$. If there is a good wind, the probability that it will fly is $\frac{5}{8}$. If there is not a good wind, the probability that the kite will fly is $\frac{1}{16}$ If the kite fly, the probability that it will stick in a tree is $\frac{1}{2}$ **SL67-73, L3 (verbal code)**

In this section, I focused on incidents where I observed a lesson integrating both verbal and nonverbal (visual and nonvisual) codes. Therefore, I referred to verbal codes as text-dominated, for example, incidents where learners were exposed to scenarios on which they had to answer questions. Nonverbal codes referred to the incidents where the teachers made use of visuals, concrete tools or diagrams to aid the teaching of probability concepts. I conclude that VNVC was moderately observed (see Table 4.17). This was so because in every lesson I observed, the teachers interchanged these codes. In this way, VNVC help contribute to quality teaching and thinking in probability lessons.

4.3.1.6 Emergent Theme: External visual presentation (EVP)

Although one expects to only see concrete objects used as visual tools to teach probability, (as had been my case in the past) during this study, I came to realize that diagrams and scenarios can also be successfully used to help the learners to think and understand the concept better (see Table 4.18). Table 4.18, therefore, shows all the external representations observed during the intervention.

Table 4.18: Excerpts of lessons relating to external visual presentation (EVP)

Mr Assery	Ms Emanu	Ms Silvy
		
		
		
		

In my opinion, it was a good idea to use external representations to teach probability because these learners are mature enough to use mental imagery to learn mathematics through experiences of critical thinking. The use of coins, dice and decks of cards could be too shallow or easy for Grade 11 learners, and in some cases, it might not add much value to their learning, compared to when they answering questions using a Tree, Venn and possibility diagrams to mention a few. In support of this, T_8 in the questionnaire also revealed that one may use graphs, charts, coins, pictures, contingency tables, posters with Venn diagrams and Tree diagrams.

4.4 CONSOLIDATION OF VISUAL PROBABILITY TEACHING STRATEGIES

In a nutshell, I conclude that all five teaching strategies were observed during the intervention period; this was a good indication of the success of the intervention. Teachers thought of adequate random events which were planned according to the various probability concepts' objectives. Some of the events were specifically planned for the introduction while others were planned for the delivery or presentation of the lesson. Even though I observed all the teaching strategies in the lessons, the first teaching strategy, IRE (see Table 4.13), requires teachers to include random events in nature, technology, and society. However, random events in technology were not observed across the lessons. Therefore, this made me think that there is a need for the teachers to learn to teach probability concepts from all three of the perspectives mentioned above (see Table 4.14).

Moreover, I observed the teachers using visual tools to teach random events and to emphasize the notion of chance. Teachers were further observed using various diagrams to facilitate learning and allow learners to explore probability concepts and expand their understanding thereof. Visuals seemed to create a friendly classroom environment. I also noticed that there was minimal work involved in lessons that required learners to use visual tools to answer probability problems, to the extent that little or no homework was given. I also acknowledge that the practical activities included the COVID-19 investigations and games, to mention a few. These seemed to create a conducive learning environment as the learners worked in groups. Similarly, they seemed to have fair discussions on the concept as they shared ideas freely and explained their reasoning to expand their understanding and knowledge. Lastly, a moderate interchange of the verbal and nonverbal codes within lessons and from lesson to the lesson was observed. Having said that, the inclusion of VNVC in probability lessons seemed to enhance mathematical thinking in probability in several ways.
From my point of view, I believe that every teacher is unique with a unique pedagogy, that enables them to manipulate and explore new ways of teaching that enhance mathematical thinking.

4.5 TEACHER ENHANCEMENT OF MATHEMATICAL THINKING PRACTICES IN PROBABILITY

In this section, I discuss how the teachers promoted mathematical thinking in their learners when they used visual teaching strategies. At times mathematics goes from being confusing, frustrating, and/or seemingly impossible, to making sense and being hard but doable. However, the ability to think mathematically is not the same as doing mathematics.

Surprisingly, , mathematical thinking includes logical and analytical thinking, quantitative reasoning and all crucial abilities (Devlin, 2012). Throughout the lesson presentations, mathematical thinking was viewed through various lenses derived from Analytical Tool B (see Appendix H). The major themes that will be discussed in this section are: Teachers' use of learners' contributions (TLIS); Encouraging multiple strategies and models (EMMS); Teacher promotion of connection making/building (PLC); Teachers addressing and utilization of learners' misconceptions (AM); Teacher facilitation of structural understanding of probability concepts (FSM).

4.5.1 Teachers' use of learners' contributions (TLIS)

Table 4.19 below shows how the teachers value and build learners' intuitive understanding of probability concepts. Throughout the lessons, the teachers fully engaged the learners through the discussions and promoted mathematical thinking through asking questions, commenting and appreciating learners' work, accommodating, revoicing and repackaging the learners' answers to expand their understanding of probability by moving the learners from the simple to complex level. Furthermore, the teachers facilitated lessons by building upon the learners' prior knowledge. Table 4.19 below shows excerpts lessons relating to the teachers' use of learners' contributions.

Table 4.19 Excerpts of lessons relating to the teachers' use of learners' contributions.

Mr Assery	Ms Emanu	Ms Silvy
▪ The probability of the girls answering the question is NOT eleven only but eleven out of sixteen. **AL3, L1** ▪ How many of you have ever seen a dice? (while showing the dice to the learners). What shape are these dice? How many possible outcomes do the dice have? **AL4, L1** ▪ Do you know the court of the arm?**AL5, L1**	If you buy twenty tickets, what are your chances of getting that prize? Learners: 20 out of 100. yes, you are right because we have 100 tickets. What are your chances of getting a prize if you buy all 100 tickets? Learners: 100%. Okay, what will be your chance of getting a prize if you did not buy any ticket? Learners: Zero, nothing. Teacher: nothing neh, because you did not buy anything. **EL10-14, L1** What do you think about the probability of getting a head? Learners: P(head)= $\frac{1}{2}$ Yes thus correct because there are only two possibilities. If I throw the dice, what is the probability of getting an even number? (she was observed throwing dice and showed the learners the outcome, which landed on a one) Learner: it is a one! How many even numbers do you see on the dice? Learners: three even numbers (2,4,6), therefore the probability is three over six that simplified to a half.**EL15-20, L1**	How likely is it to rain today? What is the probability of you waking up tomorrow and you are an albino? What is the probability that it will snow today? **SL27-28, L1**

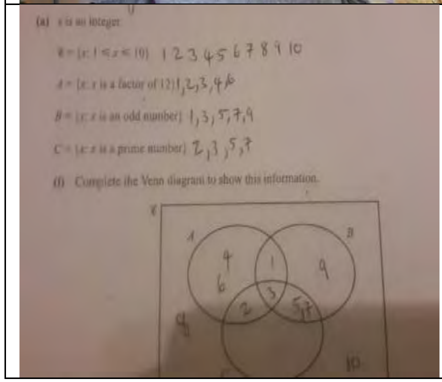
. (looking at learners' work) It is fine, why? Is it not a prime number? Aaah, Jesus, you see now? Okay, look here, use the notation to complete the statement. Group 2: Let's see, okay very technical. You see these things you start with, it is just a slight mistake, where does it belong? Group 3: Okay, you are fine, aah, come on. Yeah, this is fine, thus fine, thus beautiful, thus also okay, aah it looks nice, wuuuh, perfect, woow good. This is the only thing that was not supposed to be there, other than that, the story is nice. Wonderful. **AL36-40, L2**	Now, what will be the P(even) if the dice is thrown 100 times? Learners: 100. Firstly, you have to find the fraction of P(even) when the dice is thrown once. And then multiply that fraction by 100 that gave 50. That means if you throw a dice 100 times, the even numbers will appear 50 times. What is the P (7)? Learners: it is nothing, The answer is zero since there is no seven on the dice.**EL18-20, L2**	What is the probability of getting at least one red? What is the probability of getting two same colors? What is the probability of getting at least green? Learners were engaged fully throughout the lesson.**SL34, L2** How many are they, are they 12? How many balls do we have? What are we favoring? What do we want to get? Please stop copying now and listen. **SL33, L2**
	What is the P (pink cube)? Okay, our denominator is 10 because we have 10 cubes in total. The answer is two over ten that simplifies to one over 5. So P(blue) = three out of ten. Okay, so everyone understands how probability works now? **EL21-26, L2** [The teacher walked around observing how the learners were rolling the dice and recorded their results.] How did the game go? Does it seem fair? Who had the advantage? Why? Learners: Ms, this game is not fair, we are just getting zeros, can we start over? The game is not fair because all groups had no equal chances. **EL61-65, L3**	Okay, guys, the rule is very simple, always make sure that each branch of a tree diagram adds up to 1. As you go along the branches, you should always multiply the probabilities while when you go down the branches you should always add the probabilities. Is it clear? Is it making sense? Learners: yes, teacher. **SL67-73, L3**
	Both group 1 and 3 had only 10 out of 36 (28%) chances of winning and 26 out of 36 (72%) chances of not winning while group 2 had 16 out of 36 (44%) chances of winning and 20 out of 36 (56%) chances of not winning. One can conclude that the game was NOT fair because all groups did not have equal chances of winning. **EL21-26, L3**	Teacher: Okay let us pay attention and listen carefully.

Several excerpts in Table 4.19 reveal that the teachers were valuing their learners as they gave them a listening ear throughout the lessons. Learners were asked several questions that would lead to building their probability understanding.

4.5.2 Encouraging multiple strategies and models (EMMS)

As indicated in the teachers' extracts in Table 4.20, it is evident that teachers instructed their learners to tackle probability tasks using various approaches and models. Table 4.20 below shows how the teachers used several teaching strategies and models.

Table 4.20: Excerpts of lessons encouraging multiple strategies and models (EMMS)

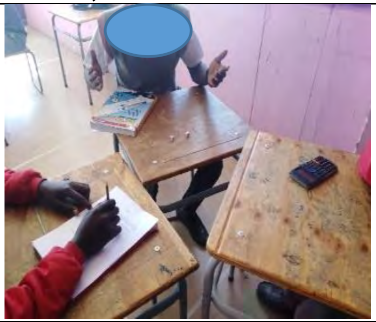
Mr Assery	Ms Emanu	Ms Silvy
The probability of a test positive result is 0.2. However, 15% of those with a positive test do not have the virus. 15% of those with negative test results do have the virus (This means that the people are given incorrect test results). Hence learners were asked to complete the tree diagram and answer the questions.**AL57-60, L3** Complete the Venn diagram in groups. ***AL36-39, L2***	Learners, you need to be extra vigilant when it comes to possibility diagrams and ask questions where you did not understand. ***EL41-44, L3***	Can you identify the prime numbers and odd numbers? Prime numbers will be circled with a yellow color while odd numbers will be circled in white colour. ***SL45-49, L2***
	I will distribute the cards to you (thirteen learners in class). Each learner got 4 different types of cards like kings, queens, jerks, clubs, red hearts, spades, diamonds, etc. She then asked the learners: How many cards are there if there are thirteen learners in the class and each learner got four cards? Learners responded that 4 x 13 = 52. The teacher: the denominator is therefore 52 which is also the total possible outcome. **EL40-44, L2**	Let us complete the tree diagram and use it to answer questions. A learner explained to the whole class and writes down the answers on the chalkboard. ***SL67-73, L3***
		

Learners were then shown to how to complete the Venn diagram, where they represented compound events and demarcated elements in the appropriate sets. Possibility diagrams were explored to find possible outcomes of combined events and analyze the patterns formed. The game approach also used a tool to promote thinking through trial and error, which enabled learners to make sense of random events and the notion of chance, and facilitate learners' understanding of probability. Tree diagrams were used to emphasize the scenario's situation and solve sequential problems or list all possibilities. All these multiple strategies helped learners to experience various approaches to tackling probability tasks and enhance their mathematical thinking.

4.5.3 Teacher promotion of connection making/building (PLC)

Lesson excerpts in Table 4.21 below shows how the teachers used models to help learners to make connections between stochastic mathematical models and strategies.

Table 4.21: Excerpts of lessons promoting of connection making/building (PLC)

Mr Assery	Ms Emanu	Ms Silvy
The probability of girls answering the question is $\frac{11}{16}$. He further instructed learners to change from $\frac{11}{16}$ vulgar fraction to a decimal fraction so that they could present it on the probability scale. ***AL3, L1***	What do you think about the probability of getting a head? **E15-17, L1** Now, what will be the P(even) if the dice is thrown 100 times? **E15-17, L1** Okay, there is something that we need to do, are you with me? Were the two red kings not included in the red cards already? Learners: yes, they were included. Okay now, what should we do? **E15-17, L1**	What is the probability of randomly choosing a red? What is the probability of not red? Ms Silvy engaged fully with the learners as they were answering questions. **S33, L1** What is the probability of getting two numbers that give a sum of 7? What is the probability of getting at least one prime or both numbers? **SL45-49, L2**
The two answers meant that the girls had a higher possibility (chances) of answering a question compared to boys in that specific class simply because there were more girls than boys. ***AL3, L1***		What is the probability of getting at least one odd? **SL48, L2** Okay, guys, the rule is very simple, always make sure that each branch of a tree diagram adds up to 1. As you go along the branches, you should always multiply the probabilities while when you go down the branches you should always add the probabilities. Is it clear? Is it making sense? **SL48, L2**
Let me say you are six boys at home and you are given two pieces of meat, the rule may not say you should divide it equally, what is the probability that you may get a piece of meat? The soup is guaranteed; it is enough for everybody. **AL6-9, L1**		
If a single dice is tossed, what is the probability that you get one? What is the probability of getting a 5? What is the probability of getting a 9? **AL4, L1** Now, our ideal version of it is you must know the possible successful outcome divided by the total possible outcome. Therefore, in this case of dice,		Learner: ohoo, so you first solve the first branch? **SL67-73, L3** If you know the probability of not flying, P (not flying) then you can get the P(flying) = …? Learner: okaaay Can you please complete another branch? Just go ahead

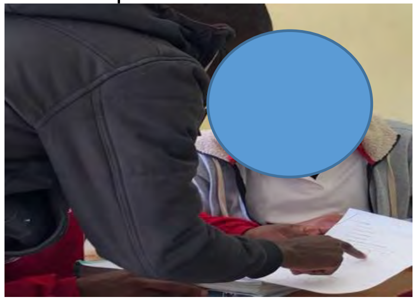
Mr Assery	Ms Emanu	Ms Silvy
the probability of getting a one will be one over six. Does it make sense to all of you? The learners easily picked up that" 15% of those tested positive do not have a virus "and "15% of those with negative result do have the virus "do not add up. It does not make sense. **AL57-60, L3** 		Learner: Okay, then you go back to the information given, you see because it says if there is a good wind, the P(kite fly) = $\frac{5}{8}$ then it comes to this branch. If it will not fly, then you go use this formula again which is 1- $\frac{5}{8}$ which is ….guys, what is the answer? Class: $\frac{3}{8}$ Can you just finish up everything? Learner: Ohoo, then here it is saying if there is no good wind and kite fly it will be $\frac{1}{16}$ **SL67-73, L3** Can you quickly explain where you are getting those answers, others also want to know. Learner: Because it is for this branch you see, if you read the statement for the information you are given, you will see here no good wind, then you check here on the branch no good wind, then you put the probability that you were given. The P (not flying)? Learner: Then you just continue again with the formula, then you say 1- $\frac{1}{16}$ = $\frac{15}{16}$ **SL67-73, L3** Teacher: Good girl, I am so excited! Thank you very much.

Table 4.21 indicates the lesson excerpts that show pushing learners conceptually. Learners were allowed to engage with mathematical models and try to make sense of their nature. At this point, learners would try to link their prior knowledge to the new knowledge and develop their understandings of probability. As a result, learners may shift from their old understandings where they believed an aspect to where they can reason and motivate their answers – which shows a sign of mathematical thinking.

4.5.4 Teachers addressing and utilising learners' misconceptions (AM)

Table 4.22 shows lesson extracts of how the teachers addressed misconceptions in learners. In this section, the teachers rectified the learners' mistakes and emphasized the concept, to ensure that learners grasped information accurately.

Table 4.22: Excerpts lessons addressing and utilizing learners' misconceptions (AM)

Mr Assery	Ms Emanu	Ms Silvy
The probability of the girls answering the question is NOT eleven only but eleven out of 16. (looking at learners' work) It is fine, why? Is it not a prime number? Aaah, Jesus, you see now? Okay, look here, use the notation to complete the statement. Group 2: Let's see, okay very technical. You see the things you start with, it is just a slight mistake, just a small one, where does it belong? **AL36-40, L2** Group 3: Okay, you are fine, aah, come on. Yeah, this is fine, thus fine, thus beautiful, thus also okay, this is the only thing that was not supposed to be there, other than that, the story is nice. Wonderful. **AL36-40, L2** you cannot be positive and yet you do not have the virus or testing negative but you have the virus" the two combinations are incorrect. **AL57-60, L3**	What is the probability of getting a five on a dice [P (5)], motivate their answers. Ms Emanu was observed throwing dice and showed the learners the outcome, which landed on a one), What is the P(even)? one learner answered: it is a one! How many even numbers do you see on the dice? Learners: three even numbers (2,4,6), therefore the probability is three over six that simplified to a half. **EL18-20, L1** Take note that whenever you throw two objects at the same time, be it coins or dice, etc, you have to draw a possibility diagram to visualise all the total possible outcomes **EL41-42, L3** All the circles were counted and they were 6 out of 36 that simplified to 1 over 6. Certain learners responded that the answer is 1 over 5 Thirty-six divided by 6 is equal to 6 **SL41-42, L2** What is the probability of getting at least a prime number, The learners focused on possibility with one prime number like (1,2 1,3 1,5), leaving the possibilities with two prime numbers like (2,3 2,5 3,3). At least in that case means it can be either one prime number or both are prime numbers.	Learner: what are the events now? Events in probability refer to, for example, getting a tail when tossing a coin is an event, rolling a 5 on a dice is an event, choosing any of the four kings is an event. Therefore, events in probability do not refer to parties like weddings or birthdays, etc **SL29-30, L1** If you throw the two dice simultaneously the total possible outcomes will be 36 (6x6) but not 12 (6+6). At least Prime? At least odd? **SL45-49, L2**

Mr Assery	Ms Emanu	Ms Silvy
	At least in that case means it can either be one prime number or both are prime numbers. If you throw the two dice simultaneously the total possible outcomes will be 36 (6x6) but not 12 (6+6). **EL41-44, L2**	
	How did the game go? Does it seem fair? Who had the advantage? Why? **EL61-65, L3**	

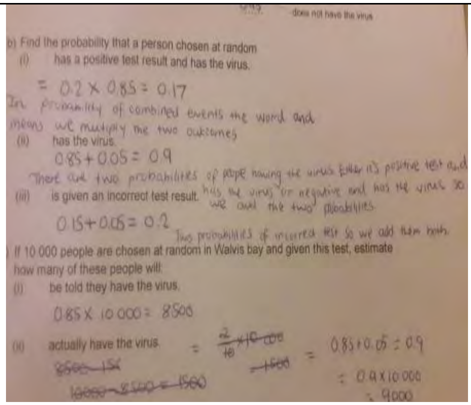
During the lessons, the teachers paid attention to their learners' answers. For every answer they got from their learners, the teachers tried to elaborate further or they asked the learners to motivate their answers. Similarly, teachers gave feedback consistently to make sure that learners corrected their mistakes. In the case where learners were doing group work, the teachers moved around the groups offering a helping hand to the learners. This was done to avoid barriers to new learning while ensuring appropriate learning in learners. By so doing, teachers promote the way learners tackle other related probability tasks.

4.5.5 Teacher facilitation of structural understanding of probability concepts (FSM)

During the observations, teachers encouraged the use of probability formulae and rules that learners had been using to work out problems. Learners could apply the probability formulae without the teacher's assistance.

Table 4.23 below shows how the teachers facilitated the learners' structural understanding of probability concepts.

Table 4.23: Excerpts of lessons showing facilitation of structural understanding of probability concepts. (FSC)

Mr Assery	Ms Emanu	Ms Silvy
	Let us present the 0 tickets, 100 tickets, and 20 tickets one on the probability scale, remember 20 tickets will be in between zero and one because it will be calculated as a fraction. **EL10-15, L1** What is the probability of getting a five on a dice [P (5)] please motivate your answers **EL10-15, L1**? Please take note probability can either be written in fractions or decimals. **EL10-15, L1** the P(red ace or black king) = $\frac{4}{52} = \frac{1}{13}$ while the P(red card or a king)=$\frac{26}{52} + \frac{4}{52} = \frac{30}{52}$. **EL41-45, L2**	She instructed the learners to identify the prime numbers and odd numbers. Okay, guys, the rule is very simple, always make sure that each branch of a tree diagram adds up to 1. As you go along the branches, you should always multiply the probabilities while when you go down the branches you should always add the probabilities. Is it clear? Is it making sense? Learners: yes, teacher **SL67-75, L1**
	Notably, the word OR in probability means addition while AND means multiplication. That is why we added the two probabilities red ace or black king, red card or a king, **EL41-45, L2**	
Change from $\frac{11}{16}$ vulgar fraction to decimal fraction so that we can present it on the probability scale. Similarly, the probability of boys answering the question in that class is $\frac{5}{16}$ that simplified to 0.3 correct to one significant figure. **AL3, L1**	Please divide yourself into groups of 3 to 4, each learner will get fair chances of throwing 2 dice simultaneously and results will be recorded. Each group will have 17 rounds to throw. **EL67-73, L3**	
The two answers meant that the girls had a high possibility (chances) of answering a question compared to boys in that specific class simply because there were more girls than boys.	Group 1 will score a point if they either get a sum of 2, 3, 4, and 5; group 2 will get a point if either they get a sum of 6, 7, and 8 while group 3 will get a point if they either got a sum of 9,10,11 and 12. **EL67-73, L3**	

Mr Assery	Ms Emanu	Ms Silvy
P(getting a piece) = $\frac{2}{6}$. Therefore, your chance of getting a piece is 0.3 which means you may run a risk of not getting a piece, your chances are minimal. But if there were 6 pieces, then you can be 100% sure that yes you can get a piece. **AL3, L1**	That further means both group 1 and 3 had only 10 out of 36 (28%) chances of winning and 26 out of 36 (72%) chances of not winning while group 2 had 16 out of 36 (44%) chances of winning and 20 out of 36 (56%) chances of not winning. She also added that thus, further means group 2 had an advantage of winning the game because it had few chances (56% compared to 72%) of losing (not winning). The game was not fair. **EL67-73, L3**	

Table 4.23 above shows excerpts of how the teachers facilitated the structural understanding of probability concepts. At this point, teachers were more interested in the structure of probability concepts. Several formulae and probability rule abbreviations were emphasized in this section. Learners were encouraged to make use of these formulae by recalling them and applying them correctly without the teachers' assistance. This helped the learners to develop some understandings of probability concepts.

4.6 SUMMARY OF THE LESSON OBSERVATIONS

In the classroom observations, it was observed in the activities presented on the chalkboard and those that were given to the learners that the use of visual tools helped the teachers to explain the concept better and more easily. This was also supported by Ms Silvy during her interview as she acknowledged that "*…visuals are very important in the sense that they help us to introduce topics better and easily because if the learner begins to work with the tools, the focus is there and the curiosity is aroused and in that regard, I look at it as a very significant exercise to execute and therefore there is no reason to discard it*".

I also noticed that visual tools helped the learners to understand the concept faster compared to how they were previously taught. Learners seemed to pay more attention and they were fully engaged. In support of this, Mr Assery during his interview also revealed that "… *visual tools do not only help the learner to process the information much better and easier but they also help the learner to concentrate on the concept that is being taught".*

There was no learner observed disturbing the class or making jokes as can happen when learners are not engaged. While this discipline and focus of learners may be attributed to a couple of factors, I concur that in these lessons, teachers' use of visuals contributed significantly to improved learner attention. The discussions and reasoning when learners were motivating their answers provided some considerably high thinking levels. Teachers used two or more concrete visual tools and for those who did not have real objects as their visual tools, they drew diagrams on the chalkboard that were used to answer questions. Learners showed a high level of interest in the lessons.

In addition, these teachers used visual tools to construct mathematical models that helped the learners to formulate images in their minds, enabling them to develop the probability ideas at hand. Platforms for classroom discussions were then created while using the mathematical models to facilitate learning. The dual use of verbal and visual teaching in these lessons helped the learners to process the information more easily and faster as they could hear and visualise as the teacher presented the lessons. Models that were used include Venn

Diagrams, contingency tables, Tree diagrams, and a probability game with a dice. Furthermore, the teachers facilitated the lessons while allowing learners to discover their learning styles and expand their thinking. As a result of the presence of visual aspects in the classrooms, learners could assess, judge and reason to agree/differ on their teachers' work on the chalkboard, and that showed understanding of the concept.

On the same note, teachers promoted mathematical thinking by fully engaging with the learners, valuing them as they listened to them attentively, and commented on their work to eliminate identified errors. Teachers also administered tasks that required various approaches and strategies, encouraging the learners to use probability formulae and rules as well as ensuring the learners' understanding of the probability concepts. Similarly, mathematical thinking was also detected through the learners' work, for example, how they approached tasks with intention of solving them, the way they employed several methods and strategies without assistance to tackle probability problems, how they identified patterns on a mathematical model, how they reasoned and the type of questions they posed for clarity.

Therefore, in my view, because of the visual tools used during the lessons, learners would assess and judge the teacher's work. Therefore, mathematical thinking was further promoted through the use of mathematical models for stochastic situations and games. Here, I have highlighted several occasions that were used to expand the learners' knowledge and enhance their mathematical thinking in probability. The next section explores the teachers' experiences of using visuals.

4.7 THE TEACHERS' EXPERIENCES OF USING VISUAL TOOLS

Individual stimulus-recall interviews were conducted with each teacher after the presentation of all three lessons, thus each teacher was interviewed once. The purpose of the interviews was to find an answer to the third research question (see Section 4.1). The interviews had seven questions based on the lessons that were taught, to seek the teachers' experiences of using visual tools in probability lessons to enhance mathematical thinking.
Question 2 of the interview asked about the importance of using visual tools when teaching probability while question 3 asked whether the teachers would continue to use visual tools to teach probability. These two questions were asking almost the same thing and therefore I decided to merge them to have one common theme (the importance of using visual tools).

The following themes emerged from the interview questions: Selection or preparation for visual tools to teach probability; the importance of using visual tools; types of visual tools used by the teachers; experience gained from working with visual tools to enhance mathematical thinking; challenges or limitations in the use of visual tools; and the new way of teaching probability after the intervention programme.

Throughout the interview process, all the teachers were very friendly and cooperative. In their responses, I noticed similarities and coherence. Therefore, responses were then merged and discussed according to the above themes.

4.7.1 Selection or preparation for visual tools to teach probability

During the interviews, teachers were probed on what informs their selection or preparation of their visual tools to teach probability. The teachers highlighted that the availability of resources as influencing their selection of visual tools. They further added that at times they use familiar tools that are easy to understand by the learners as well as readily available ones that they can bring from their homes or find on the school property like chalks, highlighters, etc. Mr Assery mentioned that:

> *…Basically to some extent what informs my selection of visual tools to teach probability is the availability of resources. Secondly, it is by some of the suggested tools in the curriculum. Sometimes I used visual tools that I have seen my colleagues using when they taught the same topic. The influence can also be based on certain things that I pick up from different sources such as watching videos on YouTube.*

Mrs. Emanu indicated that:

> *… I used a coin because at least all learners know or have money and they have used it before. Normally, kids play games with dice and they know how they look like and they can see the total possible outcomes for the coin and a dice. Any readily available materials that are familiar to my learners, something that can be understood easily can be used to introduce probability. For independent events, where we throw two dice simultaneously, it was basic to show how possibility diagrams look like and count the total possible outcome as 36 and not 12. Influence can be based on certain things you pick up from different resources such as watching videos on you-tube and you imitate how other people are doing it. Sometimes you look at the suggested tools in your curriculum/syllabus or you discuss with your head of department what tool to be used.*

Ms Silvy highlighted that***:***

> *… I used any readily available materials that are familiar to my learners, something that can be understood easily, the ones that the learners are already exposed to, and in that way, the learners are familiar with the tools.*

Familiar tools like coins are always available since learners carry their pocket money to school. Furthermore, learners play games with dice and they know how they look, which helps the teachers to introduce the lesson more easily.

From the teachers' responses above, one can deduce that teachers tried to choose visual tools that are easily accessible and familiar to their learners. However, sometimes they also look at tools suggested in the curriculum documents or check what their colleagues are using, especially during co-teaching or from YouTube videos. For their preparation of visual tools, they tended to use the tools that would make their work easier when they are introducing or delivering lessons. By so doing, this maximizes the time on the task rather than spending too much time explaining the tools.

4.7.2 The importance of using visual tools

When teachers were asked to give their views on why they think it is important to use visual tools when teaching probability and whether they will continue using visuals to teach probability, all the teachers acknowledged that visual tools are very important in the teaching of probability. Furthermore, they added that it is also equally important to use diagrams accompanied by clear explanations. In this way, visuals have the potential to enhance mathematical thinking in learners. They further added that from a psychological or cognitive point of view, learning comes in different versions: some learning takes place through seeing, touching and hearing. Therefore, to elevate learning opportunities, teachers need to maximize the possibilities in which the learners can collect or gather information from the probability lesson. Mr Assery stated that:

> *… visual tools are helpful in such a way that learners can also experience when the teacher explains, for example, the concept, of a dice. Therefore, if the learner can see, feel or touch the concrete object, then you are making maximum utilization of all the physical aspects of the learner. So it is very important to use visual tools so that we at least eliminate from our psychological understanding of mathematics, eliminate all possible barriers that can hinder the learner from processing and acquiring knowledge in the best possible manner.*
>
> *… I will continue to use visuals tools because the world is full of information and for us to enable the learners not to be distracted, we also need to understand one thing, the time we are living is psycho, learners are exposed to too many things/information and sometimes they are not so mature to learn which information/ knowledge to accept or filter. So for the teachers to helps with the distraction that comes through the learners thinking, like sometimes you are explaining the concept and the child is busy imagining*

other things in his/her mind, they need to bring in the tools that will drive the attention of the learner to the topic under discussion.

… visual tools do not only help the learner to process the information much better and easier but they also help the learner to concentrate on the concept that is being taught. Whatever the thought that might be there, when the child begins to work on the problem with the visual tools, the focus is there and the curiosity is aroused. In that regard, I look at it as a very significant exercise to execute, and therefore there is no reason to discard it.

Mrs. Emanu stated that***:***

…I think so, I will say yes, it is important for the teachers to use visual tools because learners can see something, I believe they will remember better and that will also help them in other topics when they visualise what is happening then they can give a clear or accurate answer. I think it is a good idea to continue using tools because especially now with probability lessons that I taught, the responses that I got from learners were more positive and they all understood what was going on.

Ms Silvy stated that***:***

…It is very important as a mathematics teacher to use visual tools to deliver the concept in probability. This is one of the topics that need the visual tools most because I go by the saying that goes by monkey see, monkey do. The learners learn more by seeing, feeling, and touching, it helps the learners to remember the concept thereafter a certain period. It also helps to introduce the topic better, make it easier for learners to understand the concept as well as exciting learners because it is in the form of an exercise to them as well.

I will continue using the visual tools because I think these tools will help the learners very much; it is of good advantage to the learners. It also helps the learners to process the information faster, cater the learners' attention and in that way learners are eager to learn because they can visualise, you know how playful the learners are. The way to learn through these things (visual tools).

In this way the teachers acknowledged that using visuals in teaching probability prints a permanent picture in a learner's mind and helps them to focus, visualise and remember the concepts better over a certain period. Moreover, they revealed that they believe in "monkey see, monkey do", therefore once the learners see their teachers using tools they may imitate and learn in the process. The teachers further added that visual tools excite, focus, keep the learners positive and arouse curiosity in learners. From the teachers' extracts above one can infer that visuals are crucial because they help to keep learners from distractions by drawing their attention to the topic under discussion. Hence, visuals may eliminate all possible barriers that can hinder the learning process and acquiring knowledge in the best possible manner. Visuals may also help to eliminate psychological blocks related to mathematics and help them to process information faster and more easily. Besides all this, visuals may also help teachers to introduce and deliver probability concepts appropriately and simply.

4.7.3 The types of visual tools used by the teachers

Teachers were asked to specify which visual tool among all the tools they use to help their learners to understand the content better. All teachers said that they use visuals in their teaching: coins, dice, decks of cards, coloured cubes, coloured balls and possibility, Venn and Tree diagrams that were drawn on the chalkboard during the lessons. They all pointed out that all visuals had the potential to enhance mathematical thinking in learners. However, they highlighted that each tool has its challenges. Therefore, to this question, teachers revealed that dice would be a better tool for teaching probability because it is easier for the learners to remember the content better. Dice have many possible outcomes compared to using a coin and dice can be also used to teach other topics like numbers.

Mr Assery pointed out that:

> *…I think a dice will be better compared to a coin because it has a diverse perspective but all in all the context will be the one that will determine. In general, I would say they are all important but at the expense of widening the coverage of the concept, you need to use a tool that has more diverse options like dice with 6 possible outcomes, it has different outputs that help a learner to look at it not only from the two perspectives like fora coin with two possible outcomes (head/tail). Therefore, the learner might be limited to think that the possible outcome is either a 1 or a 2 only but when you have a diverse output, a learner can think about what if this object has four faces? Will it be the same if it has 6 faces?*
>
> *So with the variety of visual tools that are there, the wider this concept, the greater might believe it is the greater the child that the child can begin to imagine. If the child's imagination grows, mathematics is a language in which we create things and it is important for us to look at it from a diverse perspective.*

Mrs. Emanu added that:

> *…I think the dice and colored cubes will help the learner to understand better compared to a coin because the coin has only two possible outcomes. For the dice, it has 6 possible outcomes and at least you can ask questions that involve the other topics of mathematics like prime numbers, even numbers, the sum of two numbers to give a 7, P (prime numbers), P (even or odd), P(kings), etc. Cubes had a total possible outcome of 10.*

Ms Silvy pointed out that***:***

> *… They are all important but just to be specific, I will choose the playing cards and dice. The dice are easier to for the kids to remember. If you draw a tree diagram it will help the learners to work out problems and give correct answers. yeah, I chose the die and the Tree diagram.*

The dominance of dice was observed during lesson because the teachers felt that dice are more familiar to the learners**.** Furthermore, cases when there are enough dice every learner, they already know what they look like and therefore explaining the concept requires less time. On the other hand, teachers prefer tools with a diverse perspective that can expose learners to more learning and higher-level thinking. Hence the type of visual tool to be used may depend on familiar tools with diverse perspectives for the sake of widening the coverage of the concept. Moreover, diagrams used as visuals in the teaching of probability may contribute to expanding the learner's thinking.

4.7.4 Experience gained after working with visual tools to enhance mathematical thinking

Teachers were asked to explain the types of experiences they gained from working with visuals after the intervention programme. They acknowledged that any type of visual tool is crucial in the teaching of probability. This is so because once visuals are integrated into probability lessons they help learners to learn differently in a positive way. Furthermore, pictures and diagrams help the learners to develop a higher level of thinking and help them to see the beauty of mathematics. They also added that other mathematical topics like sets and numbers are taught through probability. Visual tools make the lesson fun and help to control learners with unruly behaviour as it calms and focuses them on the concept being taught.

Moreover, they also noted that visual tools help learners to recall the concept faster and more easily. During the lessons, I observed that learners were quiet in class and focused on the steps used with every visual tool. Teachers also highlighted that visual tools help learners to think faster and respond to the questions as they visualise. Below are the teachers' extracts that support these statements. Mr Assery revealed that:

> *…one of the experiences I gained is the value of applying mathematical thinking outside the classroom concept. Learners need to understand that the mathematics that we teach in class is not limited to question papers and examination type of thinking only, but mathematics is designed to solve problems outside the classroom. I can recall a scenario that I used 6 boys given 2 pieces of meat. What are the chances that one*

of the 6 boys will get a piece of meat? So you can already look at it, you are using probability to see if you have two pieces of meat, maybe certain yourself that each child has to get a piece of meat, you would then be motivated to be creative and cut those pieces of meat to such an extent that they would now be able to be shared equally among the six boys that are there.

… So you are moving away from probability and again using division to apply as one of the basic operations in mathematics to ensure equitable distribution of resources. So, mathematics is not only limited to this experience, it is teaching us that we can use mathematics to solve not only mathematical problems but even social problems, political or whatever concepts that are there. Science as a matter of fact, if you look at google it was founded by a mathematician, so there are so many things that we can help ourselves to look at it. It is not a challenging subject but it is for us to think outside the box and create solutions and generally the challenges of life that we face.

… Moreover, I have realized that probability will only be a challenging topic in the view that sometimes we have perceptions that are pre-wired, old perceptions that we need to find mechanisms to overcome. Therefore, it is a challenge when one-stop thinking creatively, when you stop looking for solutions. In the same vein, it is not challenging when you start imagining and creating a new world because it allows you to look at the world from a different and diverse perspective.

Mrs. Emanu added that:

…when learners used visual tools they were more excited compared to previous lessons, visual tools made mathematics more fun and changes the attitude towards mathematics and probability to be specific. Learners could visualise and give answers promptly. I think visuals will also help learners to understand and visualise the problems in the future and solve them thereof. Similarly, visuals help learners to develop critical thinking skills, for example, dice and cards help learners to improve their probability skills.

Ms Silvy revealed that:

…the experience that I gained from using visuals is, I realized the importance of using more visuals in my teaching because I have seen that learners learn differently. When I looked at the learners that usually disturb my lessons with some jokes, this time around they were very focused and busy studying the possibility and tree diagrams. They also contributed much by answering questions compared to the previous lesson before the intervention programme. It also brought to my attention that even the learners with unruly behaviors were focusing, they were not disturbing as they use to be, they were eager to know what is happening, it seems like it is for the first time they

were experiencing such visual tools. I think it is the best thing and I will consider using it in the future.

During my presentations, I preferred drawn diagrams because I felt learners were not exposed to those types of visuals when it comes to probability but rather exposed to everyday coins, dice, and a deck of cards. I believe pictures and diagrams helped learners to create a higher level of thinking and help the learner see the beauty of mathematics. As they say "a picture is worth a thousand words". However, I am also fully supporting other visual tools that I did not use in my presentations.

The teachers revealed that visual tools themselves guide learners through to the answer when compared to working out solutions using symbols. They also noticed that visual tools increase learners' engagement and participation. Through visual manipulation, visuals might also increase learners' motivation and confidence as well as changing their mindsets. The teachers also added that visuals change attitudes towards the probability concepts and help learners to develop critical thinking. On the other hand, learners are prompted to think critically, especially when watching stimulating videos or playing games.

From the teachers' excerpts above I can conclude that visuals may be a key to constructive teaching because they encompass most, if not all, the elements of quality teaching. This includes making the lesson fun, changing attitudes towards mathematics (and probability), help to control learners with unruly behaviour, increase class participation, help learners to develop critical thinking skills, and help learners to see the beauty of mathematics by developing a higher-level of thinking. Visuals create a conducive teaching and learning environment. Conversely, the challenging probability concepts depends on the individual teacher's perspectives. Probability can be a challenging topic if one stops thinking creatively and stops looking for solutions. Similarly, probability is not a challenging topic in the sense that it allows you to look at the world through a different lens and from diverse perspectives. Therefore, this may be an eye-opener for those teachers that current view probability as a challenging topic.

4.7.5 Challenges or limitations in the use of visual tools

The three teachers revealed that there are numerous challenges associated with the use of visual tools and gave ideas on how they can be overcome. These challenges include the time to prepare/draw visual tools and the availability of adequate resources i.e teachers often buy visual tools themselves. Another challenge mentioned is that sometimes the learners got attached to the tools and instead of using them to get answers, they end up admiring and playing with them.

Mr Assery added that:

> *…One of the challenges I have experienced using the visual tools is the availability of adequate resources, at times I end up buying dices or any other possible visual tools with my own money. That forces me to buy few visual tools and in the end, we are forced to wait for all the learners to observe and have the first–hand experience with the specific tool before you introduce the concept. I took it as a challenge because it took time for learners to circulate coins and dice during my lesson. Therefore, there is a need for the science department to fully budget for the visual tools that the teachers have to use. Besides, another challenge could be some learners got attached to the tools and start playing with them. Those are the challenges that I could summarily put out in my experience.*

Mrs. Emanu:

> *…my challenge is when I have to provide the tools to all the learners. Sometimes it is just an extra cost that you have to sacrifice and buy visual tools enough for all the learners. At times when you think of preparing them yourself at home, the time is also a problem. Too many responsibilities. If the school can at least buy the visual tools once that can be kept at school to be used for a couple of years, then I don't think it will be a very big challenge for me.*

Ms Silvy:

> *… the issue of time is a challenge. It can be time to prepare the visual tools or time that visuals like a possibility, Venn, and tree diagrams took when the teacher drew them on the chalkboard. Buying from personal finance is also a challenge because at times I had to dish out money from my pocket to buy those visual tools. I think in the future; the school should also consider buying the visual tools just like it provides chalks to the teachers.*

The teacher's excerpts above reveal the challenges they encounter around the use of visual tools; however, the main challenge is the availability of adequate resources at schools. Buying the visual tools becomes expensive when there need to have enough for all the learners. At times this may demoralize the teachers from using the visual tools in their teaching of probability. Therefore, it would be of great help if visuals tools were to be included in the schools' budgets.

4.7.6 The new way of teaching probability after the intervention programme

Teachers revealed that their participation in the intervention programme made them look at visual tools from another angle. They highlighted that the old way of teaching without visuals has to change because the current learners are not the same as the learners from five or ten years ago. The teachers acknowledged that they tend to become comfortable in their teaching, sometimes ignoring/neglecting to change strategies because it is easier. In support, they emphasized the positive side of working with visual tools.

Mr Assery added that:

> *…the intervention programme changed my way of teaching probability in the sense that sometimes if you stay in the profession for so long, you think you know it all but as a new challenge comes on board, you are forced to think differently, how would I approach this? We tend to become comfortable sometimes when we are doing things, which I suspect is a psychological problem. Because as the world changes, sometimes we find it challenging at times and people do not want to take challenges. With this participation it made me look at it from a different angle than the old way where I am coming from and where I am and where we are heading is not the same world. Time has changed, so are the challenges have changed, so the approach must also change. Even the learners that we are teaching are not the same as the learners that were there 5 or 10 years ago. So, that in itself gives you an insight that it also makes me look at how to do this job differently compared to how I have been doing it in the past. This intervention, I should say it had a very constructive and positive impact.*

Ms Emanu:

> *…it changes my way of teaching because the visual tools that I used helped the learners to visualise. They also made the lesson more fun and interesting. I was also surprised that learners were given the answers so fast and there was less confusion among the learners. I believe if I continue using visual tools, learners will perform better in the future.*

Ms Silvy:

> *…Yes, I see the difference. The possibility diagrams helped me to introduce the probability concepts easily for example during my second lesson when I asked the learners what will be the total possible outcome if I throw two dice simultaneously, they responded that it will be twelve. I, therefore, drawn a possibility diagram, showed the learners all the possibilities, and further explained to the learners that the total possible outcomes are 36 (6x6) but not 12 (6+6). Learners would visualise and gave answers promptly. The intervention programme played a major role in helping the learners to understand better, therefore, I will continue using visuals to teach probability and other mathematics topics.*

The teachers revealed that the intervention programme contributed positively to their teaching. This made me think that the intervention made the teachers see the gap between their old and new ways of teaching. They also acknowledged that their teaching strategies will change for the better from now henceforth. This further means that the future is promising regarding the teaching of probability and which may shortly result in a rise in the probability performance graph.

4.8 SUMMARY OF THE INTERVIEWS

The interviews were aimed at finding answers to the third research question that focused on teachers' experiences of using visualisation tools to teach probability for promoting mathematical thinking, as a result of an intervention programme. The teachers acknowledged that visual tools are essential to the teaching and learning of probability as they enhance mathematical thinking. Either concrete objects or the diagrams drawn on the chalkboard can be used. Concerning this, T_{11} and T_{25} echoed that
visuals can be in different forms whereby

> *… one can also make use of Venn diagrams and Tree diagrams, drawing cards randomly from a deck of ten cards" to present probabilities visually*.

This further means that any type of visuals used in the teaching of probability may lead to learners deepening their understanding and expanding their mathematical thinking.

Moreover, teachers revealed that the use of visual tools helped them to introduce the concept easily in terms of saving time and effort. This made me think that once you integrate visuals into your teaching then you may need less time to explain the concepts and as a result, you may also be able to complete the syllabus within a shorter period. This may also allow you to do proper revision with the learners before they sit for their final examinations. Furthermore, the visual tools that were used seemed to engage learners more as they focused and gave their attention to the lesson. The learners were all excited to work with the tools as there was a friendly classroom atmosphere – which may have contributed to the learners constructing their knowledge as they participated in their learning.

It is said that Visual tools help the learners to understand and remember the concept better because once the learner has seen and touched the tool, it prints a permanent picture in their minds. In support of this, T_{12} revealed that learners see how ideas are connected leading to a thorough and easy understanding of new concepts as they experience concrete examples. Pictures help learners see mathematical ideas practically which facilitates higher-level thinking, and when the child participates in and has a first-hand experience with a concept, they remember more easily. The lesson of the game with dice exposed the learners to high-level thinking as they tried to identify which group had an advantage to win a game and to tell whether the game was fair or not by motivating their answers; this pushed the learners to think mathematically.

Conversely, even though the teachers echoed positive comments about visual tools, they also experienced some challenges associated with the use of visuals tools. These challenges involved the availability of resources and learners becoming attached to visuals and instead of learning, becoming distracted by them. This may be a warning light for any teacher that may be using visuals in their teaching to be extra careful and vigilant enough to consistently engage with the learners and avoid distractions. The other challenge was for the teachers to buy the visual tools using personal finances. T_2 and T_5 also acknowledged that they buy materials from personal finances and prepare the visual tools themselves like flashcards and spinners, etc.

The teachers also argued that the use of visual tools is time-consuming, for example when they do not have enough dice, coins, or cards, etc. they have to circulate them among the learners. Similarly, the time for preparing the visual tools is a challenge as sometimes the teachers are too busy and they have not enough time to prepare visual tools. Similarly, if teachers draw the Venn, possibility, and Tree diagrams on the chalkboard, by the time the learners finish copying them into their books there is not enough time to explain the concepts before the period is over. Therefore, it was suggested that schools should budget for visual tools just as they budget for chalk. However, regardless of other challenges all the teachers gave full support to the use of visual tools to teach probability and other mathematics topics.

4.9 SUMMARY OF FINDINGS

The main findings from the three sources of data: the questionnaires, classroom observations, and stimulus-recall interviews were discussed in this section. It was noted from the questionnaires that most teachers used visual tools for introducing or teaching probability. Even though a few teachers revealed that they do not use visual tools, all teachers acknowledged the importance of using such tools to enhance mathematical thinking.

Teachers argued that visual tools help the learners to understand and remember the concepts better, change their attitudes towards the concept, arouse interest and curiosity, clear misconceptions and confusion and imprint a permanent picture into the learners' memories. They further believed that as the learners use the visuals tools by touching and visualizing them, it increases their concentration, leading to them constructing knowledge as they engage with the visual tools and try to connect probability to reality.

The teachers' responses in the survey and their inadequate use of visual tools helped to craft the intervention programme. Teachers were then observed for three lessons after the two workshops of the intervention programme and the findings of the observations are summarized below. In their lessons, all three teachers used visual tools to generate images to enhance mathematical thinking. All the teachers were able to connect and support their explanations of probability to the visual tools they used. This is supported by Arcavi's (2003) definition of visualisation as:

> … The ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools, with the purpose of depicting and communicating information, thinking about and developing unknown ideas and advancing understandings. (p. 217)

The interactions between the teacher and the learners during the lessons showed that the learners were fully engaged and one could easily tell that they fully understood what the teacher was presenting. The various scenarios presented to learners were the COVID-19 situation in the Erongo Region, the issue of chance, the problem of six boys sharing two pieces of meat and the probability of winning a prize after buying tickets, pushed the learners to think more deeply about the concept. The teachers tried to connect probability to real-life situations so that the learners could think and realize that probability is not just limited to classroom concepts.

Similarly, the probability game with dice helped the learners to enhance their mathematical thinking, T_{16} revealed that dice help learners to improve their mathematics skills through mathematics mental games and learn new concepts as they play. Xiayan (2015) and Nicolson (2005) outline the significance of using games in improving the understanding of probability. Thus, the use of games could be an effective tool for facilitating learning because games motivate, challenge, increase curiosity, and promote fantasy in children (Ferguson, 2014). Therefore, introducing situations such as games in the mathematics classroom can arouse learners' interest and thus deepen their understanding of probability. Such games may also help learners to acquire an understanding of mathematical principles in probability learning as well as to gain basic knowledge, develop mathematical thinking, and acquire skills of recognizing, describing, and solving real problems by probability methods.

In support of this, Devlin (2011) states that "mathematical thinking is a whole way of looking at things, of stripping them down to their numerical, structural, or logical essentials and of analysing the underlying patterns" (p. 59). Therefore, it is about knowing how and in what way to think that provides mastery in most areas of an individual's life. Learners were asked to identify whether the game was fair and who had the advantage to win the game: this involved learners in constructive discussions that sharpened their mathematical thinking as they had to reason and motivate their answers. Concerning this, Drijvers, et al., (2019) further added that mathematical thinking also enables learners to establish a connection between theory and practice, that is, correlate theory with things they see around them regularly and acquire the habit of doing mental math, which is essential in everything that they do now, and in the future. Lastly, once mathematical thinking is fully developed, it may enable learners to solve mathematical problems in topics such as probability in a smarter way by breaking them into smaller parts and think in creative ways to arrive at a smart solution in the shortest possible time (Drijvers, et al., 2019).

Data from the interviews showed strong evidence to support the findings from the questionnaires and the classroom observations. Teachers acknowledged that the use of visual tools is the best way of disseminating knowledge. Learners were constructively busy and fully engaged in their learning. A gap was identified between the teaching of probability using visual tools and the old way of teaching without visuals. Even though visual tools are seen as time-consuming by some teachers, eventually they all supported the use of visual tools.

4.10 CONCLUSION

In this chapter, I have presented and discussed the data collected from the survey questionnaires, lesson observations, and stimulus recall interviews. Data from each research instrument has been discussed separately and concluded with a summary. The survey questionnaire data were analysed based on the themes derived from the questionnaire. The lesson observation data was based on visual probability teaching strategies and indicators for mathematical thinking. The data were further analysed using the Analytical Framework A and B (see Appendix H). Stimulus recall interview data were analysed thematically using the themes from the interview questions.

Survey data highlighted that some teachers in the Khomas region do not use visuals to teach probability and for those who used visuals, they did not use them to teach the whole lesson. There were challenges identified such as lack of resources in schools, time constraints, and teachers using their finances to buy visual tools that hindered teachers from using visual tools to teaching probability. Survey data helped to craft the intervention programme to encourage teachers to construct mathematical models. The teachers that participated in this study were therefore purposively and conveniently selected from three secondary schools.

It is evident from this study that visual strategies are imperative to teaching probability for enhancing mathematical thinking. Mathematical models constructed by the teachers during lesson observations help learners to imprint permanent images in their minds that they can reflect on to tackle probability tasks and develop the ideas for discussions in their learning about probability. Apart from visual tools used, some teachers used scenarios to link probability to real life. All teachers manipulated the models they constructed to help them facilitate the lessons. These mathematical models were in conjunction with the teachers' verbal explanations and probability terminologies that promoted mathematical thinking.

Interview findings provided strong evidence to support data from questionnaires and lesson observations. It was noticed from the interviews that teachers used visual tools to introduce and deliver the lessons. Teachers also noted that visuals attract the learners' attention, excite them and arouse their curiosity. Moreover, it was noticed from the interviews that teachers realized after the intervention that they were empowered on how to use visuals to construct mathematical models. These mathematical models help to facilitate the learning of probability concepts and for enhancing mathematical thinking. Furthermore, teachers highlighted that the old ways of teaching have to end and they need to try new teaching strategies for the betterment of teaching probability for enhancing mathematical thinking.

CHAPTER 5
SUMMARY, CONCLUSION, AND RECOMMENDATIONS

5.1 INTRODUCTION

The focus of this study was to explore how teachers use visual tools to teach probability as a result of an intervention programme. I found this study necessary because of the reported underachievement of learners in probability and the view that the use of visual tools in probability teaching has the potential to enhance learners' construction of mathematical thinking. The three selected mathematics teachers used various probability teaching strategies to teach Grade 11 mathematics learners and promote their mathematical thinking. In this study, the findings from the research were consolidated with respect to the research questions and with reference to the theoretical framework and methodological approach.

This chapter summarises the research findings and the recommendations I am making to teachers and scholars of mathematics education. It gives the limitations of the study, and further suggestions for avenues for further research. The chapter concludes with reflections on my experience of this study.

5.2 SUMMARY OF THE MAIN FINDINGS

In this section, I present the key findings according to the three research questions for this study and further summarise the findings to answer the three research questions. The summary of the findings of my study is given as follows.

5.2.1 Research question 1

How did secondary school mathematics teachers in the Khomas region teach probability in Grade 11 classrooms to promote mathematical thinking before the intervention programme?

Thirty-four teachers who participated in the survey questionnaires revealed that they use visual tools to teach probability. Most of their answers showed that they are aware of various possible visual tools. The teachers indicated that they use different types of visuals like concrete materials, external representations, games and YouTube videos. They also revealed that they prefer to use materials that are familiar to the learners and readily available.

The survey further revealed that although the teachers in the Khomas region acknowledged the significance of using visual tools in teaching probability, most of them were either using them occasionally or did not use them at all. Those who used some visuals, did not use them throughout their lessons. The majority of the respondents highlighted that the use of visual tools in the teaching of probability is the best way to disseminate knowledge.

Similarly, their responses further showed that there is a great significance in using visuals with a challenging topic such as probability. Visuals arouse interest and curiosity, develop high-order thinking and enhance tactile hand-eye-mind connections that improve the ability to recall facts/concepts to retain probability skills. Therefore, using manipulatives helps learners with their limited maturity, to draw their learning from concrete materials in order grasp abstract mathematical concepts (Piaget, 1952).

Respondents asserted that visual tools open learners' minds and help them to think more deeply and faster. It is also noted that visuals improve lessons by making them lively and help the teacher to explain the concept more easily as the learners learn by doing (touching and seeing the real objects) rather than mere symbols or theory. Additionally, seeing and touching clears up misconceptions and confusion, and leaves/imprints a clear picture in the person's mind to enhance a broader and more realistic understanding. Cockett and Kilgour (2015) argued that the use of manipulatives helps learners to reason mathematically by aiding them to construct a meaningful understanding of abstract concepts.

Teachers also revealed their challenges associated with teaching probability, which include the inadequate resources at schools, time for preparing or drawing visual aspects on the chalkboard, using their finances to buy the visual tools, and lastly, learners becoming attached to the tools by admiring them instead of using them for learning. These challenges mentioned above could be the reason why some teachers do not use visuals to teach probability concepts. However, although a few respondents acknowledged that they do not use visuals, they fully support the use of visuals as they believe that it promotes mathematical thinking.

5.2.2 Research question 2

How do selected teachers use visualisation tools to promote mathematical thinking during the teaching of probability concepts in Grade 11 classrooms as a result of participating in an intervention programme?

The participating teachers used visual strategies to promote mathematical thinking about probability during their teaching in numerous ways. These visual strategies therefore categorized into five themes as indicated on the Analytical Tool A (see Appendix H). These include: Identifying random events in nature, technology, and society; analysing conditions of random events and derive appropriate modelling assumptions; constructing mathematical models for stochastic situations and explore various scenarios and outcomes from these models; applying mathematical methods and procedures of probability; and lastly, the integration of verbal and nonverbal codes in probability lessons.

I therefore, discussed the themes in the order that I have listed them above. Because the first two themes are interrelated and they share a lot in common, I merged them and presented their findings together. The random events observed across the lessons were identified and analysed according to their type as nature and society. Random events in nature were classified by looking at the aspects of nature or anything that has to do with nature, for example genetics, rain, snow, etc. while random events in society looked at any situation that happens in our daily lives in our communities and things that we are mostly exposed to. Throughout all the lessons, there were no technology random events observed. The modelling assumptions for the observed random events were to teach possible outcomes of single events which included tossing coins, picking coloured cubes from a bag and throwing dice, and present them on the probability scale. Possible outcomes for simple combined events included the use of Venn diagrams, Tree diagrams contingency tables and decks of cards and lastly the teaching of probability in practice involved real-life scenarios and games.

Teachers prepared visual tools for the mathematical models to create images that helped to develop the probability ideas at hand. These models were used throughout the lesson presentations. Two teachers used contingency tables to teach simple combined events while the other teacher used the Venn diagram. The teachers, thus used these models to create platforms for classroom discussions that were driven by questions aiming to check the learners' prior knowledge and for assessing whether they understood the concept under discussion. The teachers facilitated the use of contingency tables by asking the learners to find the P (sum of 7), P (at least prime), P (at least odd), among others. Learners also identified patterns that were formed by various answers. Venn diagrams were used to demarcate elements of sets that were given in set builder notation and calculate the P (odd and prime), P (two odd), to mention a few. The mathematical models showed a correlation with the visual tools prepared. Because of the visual aspects drawn on the chalkboard, learners could assess and judge their teachers' work. These models drew the learners' attention and interest as well as engaging them actively as they visualised the tools. For this reason, one may infer that learning for understanding was taking place.

During the lesson observations, the teachers facilitated the learning, assisting learners to connect probability to real-life situations by using various scenarios. The teachers administered group work where the learners were motivated to discuss and explain to others how they got their answers. Furthermore, the learners were also engaged in probability games with dice to emphasize the notion of random. They were exposed to various methods and procedures of solving probability problems. The teachers involved the learners fully in order to help them understand the concept and enhance their mathematical thinking in the process.

The use of scenarios, concrete materials, and external representations in teaching probability was in conjunction with the teachers' explanations and probability terminologies. They encouraged their learners to use probability formulae and rules consistently. Therefore, there was fair integration of both verbal and visual codes throughout the lessons.

During this study, the use of external representations was observed moderately. Teachers used various diagrams like Venn diagrams, Tree diagrams, and contingency tables resulting from the various scenarios being discussed to add more value to the learning. These representations have an unofficial list of ways to present the problem and its solution that expresses thinking in a variety of ways, for example, words, drawings or pictures, charts, graphs, and written explanations. Representations help show exactly what a learner is thinking when figuring out a problem and arriving at a solution (Lukáč & Gavala, 2019).

During the lessons, mathematical thinking was identified through the themes from the Analytical Tool B (see Appendix H), Learning to think mathematically is not about getting answers Devlin (2012), though once you have learned to think mathematically, getting the right answer becomes a lot easier than when you are just following procedural recipes. Therefore, mathematical thinking may be viewed as a process of trying and reflecting but not getting answers.

I observed that all the teachers value their learners. This was so because all the teachers gave their learners a listening ear. In most lessons, if not all, the teachers tried to give fair opportunities to answer questions to all the learners. This was done so that learners did not all talk at the same time, to give time to others to think about a probability problem and listen to each other to learn in the process. Similarly, teachers also tried to give prompt feedback and comment on the learners' answers to emphasize the concept, minimise the identified errors, and expand the learners' mathematical thinking.

Teachers instructed learners to answer various activities with different approaches. These activities required learners to draw the Venn diagrams, use the contingency tables, and complete the Tree diagrams to answer the questions. The games approach was also acknowledged. The use of multiple strategies and models resulted in learners expanding their knowledge and enhancing their mathematical thinking in the process.

Contingency tables were used to facilitate the possible outcomes for simple combined events. Teachers helped the learners to identify the patterns that were formed by various probability answers as they formed the diagonals. In the lesson that used the games approach, the teacher allocated enough time for learners to explore the game. The learners were asked the questions to emphasize their understanding of random events and help them to deepen their understanding and enhance their thinking.

Throughout the lessons, the teachers gave attention their learners' answers, therefore, most, if not every answer that was given by the learners, was extended to ensure that the learners grasped the concept effectively. Furthermore, the teachers helped the learners within their groups to eliminate errors/mistakes. Similarly, for the games approach, the teacher explained the chances of winning with random events in detail to ensure understanding.

During my observations, the teachers emphasized the use of probability formulae and rules. Because they consistently emphasized these formulae, rules, and notations, learners could apply them correctly to solve probability problems. Having said that, teachers used visual tools to construct mathematical models, thus helping learners to create images that enabled them to develop the probability ideas at hand. The type of visuals that were used to construct models included scenarios, concrete materials and diagrams. These mathematical models were then used throughout the lessons to create platforms for classroom discussions driven by several questions related to the mathematical models being used. Furthermore, contingency tables were used to find the total possible outcomes for simple combined events and to develop patterns. Venn diagrams were used to emphasize set builder notation, drawing sets and calculate probability questions while Tree diagrams were used to emphasize the rule and formulae for probability. In support of this, Devlin (2011) states that "mathematical thinking is a whole way of looking at things, of stripping them down to their numerical, structural, or logical essentials and of analysing the underlying patterns" (p. 59). Because of the presence of visual models in probability lessons, learners could assess, judge and take part in friendly debates over their teachers' work and this resulted in fruitful discussions that promote mathematical thinking.

5.2.3 Research question 3

What are the selected teachers' experiences of using visualisation tools to teach probability for promoting mathematical thinking as a result of an intervention programme?

To answer this question, I looked at the teachers' experiences through the themes that I generated from the interview questions. The second and third themes were interrelated and therefore, they were merged.

The teachers tried to use visuals that were easily accessible and familiar to their learners. This was so because the tools made their work much easier when they introduced and delivered the lessons. By so doing, the teachers maximized on the time on task for the learners to explore the probability concepts, and hence could focus on enhancing learners' mathematical thinking.

All the teachers acknowledged that all types of visuals are important in the teaching of probability because they all have the potential to enhance mathematical thinking in learners. They further added that learners learn in different ways, therefore to elevate the learners' learning opportunities, teachers need to maximize the possibility in which learners can gather information from probability lessons and use it to expand their ways of thinking and deepen their understanding. They also highlighted that through the use of visuals, learners may be eliminated from their psychological blocks related to mathematics and help them to process the information faster and more easily.

The teachers chose dice rather coins because they have many outputs. Similarly, the teachers preferred a tool with diverse perspectives that can expose learners to more learning and enable high-level thinking in learners. Therefore, the type of visuals may depend on familiar tools with diverse perspectives for the sake of widening the coverage of the concept while expanding the learners' thinking.
The teachers acknowledged that any type of visuals is crucial to help promote mathematical thinking. Concrete materials and diagrams help the learners to see the beauty of mathematics. Other mathematical topics like sets, numbers, or gradients may also be taught through the use of probability models by identifying patterns, thus allowing learners to appreciate the beauty of mathematics. Through using visuals in probability lessons, learners may change their mind-sets and attitudes towards probability concepts and develop critical thinking. The main challenge that the teachers have is the inadequate resources at schools, the time constraints, and learners ending up admiring the tools instead of using them for learning.

The teachers acknowledged that the intervention programme made them realize the gap between their old ways of teaching and new way they experienced on the intervention programme. This further means that teachers acknowledged the change for better when using visual teaching strategies to teach the concept of probability.

In addition, the use of visuals enabled the teachers to facilitate the lessons and aid learners in constructing their knowledge. This further enabled the learners to build on their prior knowledge and make sense of probability concepts. Similarly, the visuals employed by the teachers helped to extend the concepts to a deeper level to ensure deeper understanding. Since learners have different learning abilities, the teachers used visuals to promote the use of different senses. Thus, learners could observe and hear as the teacher demonstrated. This strategy may also aid learners to process information equally into their working memories as the cognitive process is done over verbal and visual channels. Lastly, the teachers highlighted that the old ways of teaching have to end and new visual teaching strategies need to be implemented, for the betterment of teaching probability concepts and for enhancing mathematical thinking. The findings of the study showed that the teachers expanded their knowledge of teaching probability concepts to enhance mathematical thinking through the use of visual tools and mathematical models.

5.3 RECOMMENDATIONS

This study was focused on the visual probability teaching strategies to enhance mathematical thinking. From the findings of this study, I suggest the recommendations below.
I believe that traditional methods are deemed to be too deductive, therefore, mathematics teachers should integrate visuals in their probability lessons and actively involve learners to improve their learning pace and enhance mathematical thinking. Teachers should make and/or be provided with enough visual tools to enable learners to manipulate them in groups instead of the drawing them during the lessons: this would minimize the time spent on drawing. Teachers should also emphasize the application of probability rules and formulae as well as expose learners to probability terminologies such as “random’, ‘chance’, ‘likelihood’ among others. Similarly, learners should be afforded an opportunity of exploring the visual tools such as dice, spinners and decks of cards to help them to create images in their minds which they can later reflect on and use to solve probability tasks with the teacher facilitating the process. Teachers, therefore, need to familiarize themselves with visual models such as contingency tables, Tree diagrams, Venn diagrams etc. before the lesson, to avoid frustrations in both the teacher and learners. There is a need for teachers to consider using scenarios to formulate models in creating a platform for discussion, to ensure that their verbal explanations are in line with the visuals being used. Using visual strategies aid learners to solve probability problems easily and correctly. Therefore, subject advisors should also regulate the provision of refresher

workshops for teachers and encourage co-teaching within educational clusters, circuits, and regions. On the same note, teachers also need to train their learners on how to use the models and strategies, and also encourage them to use such strategies when solving probability tasks. In the case where mathematics teachers have to create or develop their teaching materials, they should create attractive and educational visuals related to the topic, and encourage the use of both concrete materials like spinners and flash cards and external representations like Tree diagrams. They should also use different methods and approaches to teach probability and mathematics topics generally. Similarly, learners should be given opportunities to design visual tools and use during their lessons for assessment purposes. This may enable the learners to apply their knowledge to solve mathematical problems and enhance their mathematical thinking and skills.

Based on the evidence of the visual tools used in this study, I therefore recommend that the Namibian mathematics curriculum should be visually oriented, by emphasizing visualisation tools to enhance mathematical thinking. With mathematics being compulsory in the Namibian curriculum for all learners, visual tools may help the less-gifted learners to understand probability concepts. In addition, this study learned much about indicators for visual teaching strategies on probability and mathematical thinking, and suggests that teachers are exposed to these types of teaching strategies.

5.4 CONTRIBUTIONS OF THE STUDY

This study expanded my knowledge about visualisation in mathematics education and how to use effective visual teaching strategies in probability concepts. I hope that these findings will inspire mathematics teachers, particularly in the Khomas region, Circuit 3 where the study took place, and contribute towards improving the compulsory mathematics curriculum, emphasizing the important role of visualisation to enhance mathematical thinking in learners. These findings may further encourage and strengthen the use of visual tools in probability lessons in Grade 11 classrooms.

I believe in the powerful maxim that says “where there is a will, there is a way”. Therefore, once visual tools are used effectively, they have the potential to elicit a positive learning environment that facilitates teaching and increases learners’ curiosity and interest in the learning of probability and mathematics in general.

5.5 LIMITATIONS

Just as other studies encounter challenges and limitations, this study was no exception. The limitations and challenges for this study were as follows: The fact that I am a full-time teacher and that all secondary schools in the Khomas region finish at 13h00, it was hard for me to collect the questionnaires from the teachers who could not submit theirs on time with their circuit representatives. These findings might not be generalized because of the exploratory nature of my study that involved only three teachers from three schools who were purposively and conveniently selected. Similarly, the lockdown of schools for three consecutive months due to the COVID-19 pandemic (that struck the whole world, including Namibia), resulted in a prolonged period of data collection, interfering the with the smooth flow of the study and affecting the teachers' willingness and ability to prepare well in advance for their probability lessons.

The study focused on how the teachers teach probability using visual tools for enhancing mathematical thinking in learners. However, a fuller picture of enhancing mathematical thinking in learners could have been noticed or found if the study consisted of considerations from the learners of the participating teachers.

5.6 SUGGESTIONS FOR FURTHER RESEARCH

Based on the findings of this study, the following are suggested as areas for further research: The population for this study was fairly small, situated in one region and with three secondary school teachers; this means the findings for this study cannot be generalized to the entire population of learners in other regions and the country at large. There is a need to extend the study to a bigger sample of teachers from a variety of schools to determine the use of visuals in the general population. This would enable the study to be generalized and used in a wider context to improve classroom practice and teach probability to enhance learners' mathematical thinking.

The study focused specifically on the Grade 11 mathematics teachers on how they teach probability using visual tools for enhancing learners' mathematical thinking. However, further research should be conducted to examine the use of visuals in other mathematics topics or other subjects at all grade levels.

5.7 PERSONAL REFLECTIONS

I stood by the beliefs “winners never quit” and “where there is a will, there is a way”. These two beliefs and the fear of being a university dropout kept me going throughout my research journey. I should confess that “the struggle to come up with this thesis was for real”. Had it not been for the excellent supervisors that coached and guided me tirelessly to pull this research wagon, then reaching this point could have been an unaccomplished dream. Nevertheless, it was a great privilege to be involved in a visualisation research project for the teaching and learning of mathematics and specifically, probability.

Mathematics was never one of my best subjects in primary school. I developed a love for mathematics at secondary school because I was inspired by my mathematics teacher who was also then our school principal. Being a researcher in visualisation processes greatly improved my knowledge and understanding of various approaches in mathematics. This research was stimulated by my desire and motivation to contribute to the development of Namibia’s education system.

The engagements I explored during the journey of this research awarded me an experience in understanding research design and methodologies. Interestingly, the writing up of this thesis has improved my reading and writing skills tremendously. Despite all the challenges I encountered during my research period, I had a great passion and desire to learn and obtain my Master’s degree one fine day. I see myself as a future mathematics researcher, planner, and educator.

5.8 CONCLUSION

It is evident from this study that visual teaching strategies are imperative in the teaching of probability. It is, therefore, noted that using visual models to teach probability helps learners to imprint permanent pictures in their minds, which they use to understand and solve these problems.

In this chapter, I concluded the whole research process by presenting a summary of the main findings of the study and contributions of the study, discussed the limitations of the study, and made recommendations that arose from the findings. I also shared my reflections on the whole research exercise.

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APPENDICES

APPENDIX A: ETHICAL CLEARANCE LETTER FROM RHODES UNIVERSITY

RHODES UNIVERSITY
Where leaders learn

Human Ethics subcommittee
Rhodes University Ethical Standards Committee
PO Box 94, Grahamstown, 6140, South Africa
t +27 (0) 46 603 8055
f +27 (0) 46 603 8822
e ethics-committee@ru.ac.za

www.ru.ac.za/research/research/ethics
NHREC Registration no. REC-241114-045

30 April 2020

Lavinia Tangi-Jehova Nghidinwa

Email: g19n9934@campus.ru.ac.za

Review Reference: 2020-0812-3325

Dear Dr. Chikiwa

Title: Exploring visual probability teaching strategies for enhancing mathematical thinking in grade 11 classes.

Principal Investigator: DR Clemence Chikiwa

Collaborators: Mrs. Lavinia Tangi - Jehova Nghidinwa,

This letter confirms that the above research proposal has been reviewed and **APPROVED** by the Rhodes University Ethical Standards Committee (RUESC) – Human Ethics (HE) sub-committee.

Approval has been granted for 1 year. An annual progress report will be required in order to renew approval for an additional period. You will receive an email notifying when the annual report is due.

Please ensure that the ethical standards committee is notified should any substantive change(s) be made, for whatever reason, during the research process. This includes changes in investigators. Please also ensure that a brief report is submitted to the ethics committee on the completion of the research. The purpose of this report is to indicate whether the research was conducted successfully, if any aspects could not be completed, or if any problems arose that the ethical standards committee should be aware of. If a thesis or dissertation arising from this research is submitted to the library's electronic theses and dissertations (ETD) repository, please notify the committee of the date of submission and/or any reference or cataloging number allocated.

Sincerely,

Arthur Webb

Prof Arthur Webb

Chair: Human Ethics Sub-Committee, RUESC- HE

APPENDIX B: PERMISSION LETTER FROM THE DIRECTOR OF EDUCATION KHOMAS REGION

REPUBLIC OF NAMIBIA

KHOMAS REGIONAL COUNCIL

DIRECTORATE OF EDUCATION, ARTS AND CULTURE

Tel: [09 264 61] 293 4356
Fax: [09 264 61] 231 367/248 251

Private Bag 13236
WINDHOEK

28 February 2020

Rhodes University
Drosty Road
Grahamstown 6139
Contact: 0813776668

For Attention: Ms Lavinia T – J Nghidinwa

REQUEST FOR PERMISSION TO CONDUCT RESEARCH INTERVIEWS WITH SELECTED SECONDARY SCHOOLS IN KHOMAS REGION

Your letter dated 28 February 2020 on the above topic is hereby acknowledged.

Permission is hereby given to you to collect data on "Exploring visual teaching strategies for enhancing mathematical thinking in Grade 11 probability classes" at Acacia High School, Eldorado Secondary School and Ella Du Plessis High School in Khomas Region under the following conditions:

- The Principal of the selected school to be visited must be contacted in advance and agreement should be reached between you and the Principal.
- The school programme should not be interrupted.
- The teachers and learners who will take part in this exercise will do so voluntarily.
- The Directorate of Education, Arts and Culture should be provided with a copy of your thesis/ findings.

We wish you success in your research.

Yours sincerely

28/02/2020

MINISTRY OF EDUCATION
ARTS AND CULTURE
PRIVATE BAG 13236 WINDHOEK
28 -02- 2020
DIRECTOR
KHOMAS REGION

Gerard N. Vries
Director of Education, Arts and Culture

Page 1 of 1

APPENDIX C: PERMISSION LETTERS FROM THE PRINCIPALS: SCHOOL A

ACACIA HIGH SCHOOL

[redacted] **HIGH SCHOOL**

P.O.Box 10130 - Khomasdal - Tel: 210887 - Fax: 213481 - E-mail: acaciahigh@iway.na

22 June 2020

Dear Ms. Lavinia T. Nghidinwa

RE: PERMISION TO CONDUCT A RESEARCH

This communique serves to confirm that your requisition to conduct a research on "Exploring visual probability teaching strategies for enhancing Mathematical thinking in Grade 11 classes[redacted] High school has been granted on condition that:

1. Your research may not interfere with the school programme.
2. Participants to your research project may not be coerced in any form to participate.

Kindly share with us your research findings in order for us to introspect and learn from our shortcomings.

Yours faithfully

[redacted]tment

[redacted]ATION
[redacted]CHOOL
2020 -06- 22
P.O. Box 10130, Khomasdal
Windhoek
Tel: 061 - 210887
REPUBLIC OF NAMIBIA

APPENDIX C: PERMISSION LETTERS FROM THE PRINCIPALS : SCHOOL B

[redacted] **SECONDARY SCHOOL**

P.O BOX 10547
KHOMASDAL

TEL.: 061 - 213380
TEL.: 061 - 212543
FAX: 061 - 212559
E-mail: eldorado@iway.na

29th June 2020

Lavinia T. Nghidinwa
Student
Rhodes University
Republic of South Africa

RE: PERMISSION TO CONDUCT RESEARC[redacted] SECONDARY SCHOOL, KHOMAS REGION.

1. I acknowledge the receipt of your letter dated 22nd June 2020 and therefore it bears reference;

2. Kindly be informed that the permission is hereby granted to conduct the study entitled: ***Exploring visual probability teaching strategies for enhancing mathematical thinking in grade 11 classes.***

3. This permission is subject to the following strict conditions, (i) there should be minimal or no interruption on normal teaching and learning, during a class. (ii) Ethical issues of confidentiality and anonymity should be respected and retained throughout this activity i.e voluntary participation, and consent from participants.

4. Furthermore, we humbly request you to share with us your research findings.

5. I wish you the best in conducting your study,

[redacted] EDUCATION
SECONDARY SCHOOL
2020 -06- 29
P.O. Box 10547, Khomasdal
Tel: 061-212543
Republic of Namibia

Yours faithfully

....
Mrs PRINCIPAL (Principal)

APPENDIX C: PERMISSION LETTERS FROM THE PRINCIPALS: SCHOOL C

ELLA DU PLESSIS HIGH SCHOOL

PER ARDUA AD ASTRA

[redacted] High School

[redacted] / Fax:061 213 777 Cell No:081 382 1680

elladuplessis@yahoo.com www.elladuplessis.com

01 July 2020

Lavinia T. Nghidinwa
Student
Rhodes University
Republic of South Africa

SUBJECT: PERMISSION TO CONDUCT EDUCATIONAL RESEARCH AT [redacted] HIGH SCHOOL

The above subject matter refers.

I, [redacted] Principal of [redacted] School hereby grant you permission to [redacted] fulfilment of your Master degree in Mathematics Education. The following teachers will assist you: Ms. Nghidengwa and Ms. Kambala.

Hope and trust that they will assist you without any problem.

Yours,

[redacted]

0 1 JUL 2020
PO Box 10399-Khomasdal
Tel: 061-211307·Fax: 061-213777
E-mail: elladuplessis@yahoo.com
DIRECTORATE OF EDUCATION

[redacted]

PRINCIPAL

All correspondence must be addressed to the Principal

APPENDIX D: INFORMED CONSENT FROM THE PARTICIPANT

PARTICIPANT INFORMED CONSENT

INFORMED CONSENT DECLARATION

(Participant)

Project Title: Exploring visual teaching strategies for enhancing mathematical thinking in grade 11 probability classes.

(My name is Ms. Lavinia Tangi – Jehova Nghidinwa) from the Department of Education, Rhodes University has requested my permission to participate in the above-mentioned research project.

The nature and the purpose of the research project and of this informed consent declaration have been explained to me in a language that I understand.

I am aware that:

1. The purpose of the research project is to determine how the use of visualization tools in the teaching of probability can promote mathematical thinking at the secondary level.

2. The Rhodes University has given ethical clearance to this research project and I have seen/ may request to see the clearance certificate.

3. By participating in this research project I will be contributing towards the improvement in the teaching of probability using different visualisation tools. Visualisation refers to the use of visual models to improve the teaching and learning of different concepts in mathematics, including probability. Since learners struggle to understand probability, exploring different visual teaching strategies could be beneficial for our learners.

4. I will participate in the project by being an active member of an intervention programme for this study, I will find the activities involved in this study quite familiar since I will be involved in the development and use of visual materials. It is against this backdrop that I have been selected as one of the participants in this study.

5. My participation is entirely voluntary and should I at any stage wish to withdraw from participating further, I may do so without any negative consequences.

6. I will not be compensated for participating in the research, but my out-of-pocket expenses will be reimbursed.

7. There may be risks associated with my participation in the project. I am aware that

   a. the following risks are associated with my participation: confidentiality and anonymity.
   b. the following steps have been taken to prevent the risks: Issues of confidentiality and anonymity will be addressed by using pseudonyms rather than the real names of schools and teachers involved in this study.
   c. there is a 40 % chance of the risk materialising

8. The researcher intends publishing the research results in the form of à thesis, However, confidentiality and anonymity of records will be maintained and that my name and identity will not be revealed to anyone who has not been involved in the conduct of the research.

9. I will receive feedback in the form of thesis, workshops, and journals regarding the results obtained during the study.

10. Any further questions that I might have concerning the research or my participation will be answered by Ms. Lavinia Tangi – Jehova Nghidinwa at 081 3776668.

11. By signing this informed consent declaration, I am not waiving any legal claims, rights or remedies.

12. A copy of this informed consent declaration will be given to me, and the original will be kept on record.

13. Request to take pictures, video and voice recording for this study

I, ...[redacted].......... have read the above information / confirm that the above information has been explained to me in a language that I understand and I am aware of this document's contents. I have asked all the questions that I wished to ask and these have been answered to my satisfaction. I fully understand what is expected of me during the research.

I have not been pressurised in any way and I voluntarily agree to participate in the above-

[redacted]. 09 · 03 · 21............

Participants signature **Witness** **Date**

Rhodes University, Research Office, Ethics
Ethics Coordinator: ethics-commitee@ru.ac.za
t: +27 (0) 46 603 7727 f: +27 (0) 86 616 7707
Room 220, Main Admin Building, Drostdy Road, Grahamstown, 6139

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13. Request to take pictures, video and voice recording for this study

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I have not been pressurised in any way and I voluntarily agree to participate in the above-mentioned project.

[redacted] [redacted] ......5.04.21......

Participants signature **Witness** **Date**

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PARTICIPANT INFORMED CONSENT

INFORMED CONSENT DECLARATION

(Participant)

Project Title: Exploring visual teaching strategies for enhancing mathematical thinking in grade 11 probability classes.
(My name is Ms. Lavinia Tangi – Jehova Nghidinwa) from the Department of Education, Rhodes University has requested my permission to participate in the above-mentioned research project.

The nature and the purpose of the research project and of this informed consent declaration have been explained to me in a language that I understand.

I am aware that:

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2. The Rhodes University has given ethical clearance to this research project and I have seen/ may request to see the clearance certificate.

3. By participating in this research project I will be contributing towards the improvement in the teaching of probability using different visualisation tools. Visualisation refers to the use of visual models to improve the teaching and learning of different concepts in mathematics, including probability. Since learners struggle to understand probability, exploring different visual teaching strategies could be beneficial for our learners.

4. I will participate in the project by being an active member of an intervention programme for this study, I will find the activities involved in this study quite familiar since I will be involved in the development and use of visual materials. It is against this backdrop that I have been selected as one of the participants in this study.

5. My participation is entirely voluntary and should I at any stage wish to withdraw from participating further, I may do so without any negative consequences.

6. I will not be compensated for participating in the research, but my out-of-pocket expenses will be reimbursed.

7. There may be risks associated with my participation in the project. I am aware that

   a. the following risks are associated with my participation: confidentiality and anonymity.
   b. the following steps have been taken to prevent the risks: Issues of confidentiality and anonymity will be addressed by using pseudonyms rather than the real names of schools and teachers involved in this study.
   c. there is a 40 % chance of the risk materialising

8. The researcher intends publishing the research results in the form of à thesis, However, confidentiality and anonymity of records will be maintained and that my name and identity will not be revealed to anyone who has not been involved in the conduct of the research.

9. I will receive feedback in the form of thesis, workshops, and journals regarding the results obtained during the study.

10. Any further questions that I might have concerning the research or my participation will be answered by Ms. Lavinia Tangi – Jehova Nghidinwa at 081 3776668.

11. By signing this informed consent declaration, I am not waiving any legal claims, rights or remedies.

12. A copy of this informed consent declaration will be given to me, and the original will be kept on record.

13. Request to take pictures, video and voice recording for this study

I, ...[redacted]...................................... have read the above information / confirm that the above information has been explained to me in a language that I understand and I am aware of this document's contents. I have asked all the questions that I wished to ask and these have been answered to my satisfaction. I fully understand what is expected of me during the research.

I have not been pressurised in any way and I voluntarily agree to participate in the above-me[redacted]t.

....[redacted].................	....[redacted].....	09.03.21.........
Participants signature	**Witness**	**Date**

Rhodes University, Research Office, Ethics
Ethics Coordinator: ethics-commitee@ru.ac.za
t: +27 (0) 46 603 7727 f: +27 (0) 86 616 7707
Room 220, Main Admin Building, Drostdy Road, Grahamstown, 6139

APPENDIX E: INFORMED CONSENT FROM THE LEARNER

CHILD PARTICIPANT'S ASSENT FORM

INFORMED CONSENT

DECLARATION

(Child participant)

Project Title: Exploring visual teaching strategies for enhancing mathematical thinking in grade 11 probability classes.
Upon the completion of this study, learners will be provided with the research findings via their mathematics teachers and guardians upon request. data will be stored for 5 years in an external drive and clouds that will be locked (password protected) in a way that is hard to access. A back up copy will be saved on google account. This study is being done under the auspices of the Mathematics Chair.

Researcher's name: Ms. Lavinia Tangi – Jehova Nghidinwa
For enquiries contact the researcher at 081 377 6668, Email: tangijehova@yahoo.com or **Dr. Clemence Chikiwa** (supervisor), Tell: +27 46 603 7210, Email: c.chikiwa@ru.ac.za.

Name of participant: [redacted]............

1. Has the researcher explained what s/he will be doing and wants you to do?

YES ✓ NO

2. Has the researcher explained why s/he wants you to take part?

YES ✓ NO

3. Do you understand what the research wants to do

YES ✓ NO

4. Do you know if anything good or bad can happen to you during the research?

YES ✓ NO

5. Do you know that your name and what you say will be kept a secret from other people?

6. Did you ask the researcher any questions about the research? YES NO

7. Has the researcher answered all your questions? YES NO

8. Do you understand that you can refuse to participate if you do not want to take part and that nothing will happen to you if you refuse? YES NO

9. Do you understand that you may pull out of the study at any time if you no longer want to continue? YES NO

10. Do you know who to talk to if you are worried or have any other questions to ask? YES NO

11. Has anyone forced or put pressure on you to take part in this research? YES NO

12. Are you willing to take part in the research? YES NO

[redacted] 01 June 2020
Date

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APPENDIX F: INFORMED CONSENT FROM THE PARENT

RHODES UNIVERSITY
Where leaders learn

PARENT AND GUARDIAN'S INFORMED CONSENT

INFORMED CONSENT DECLARATION
(Parent or Guardian)

Project Title: Exploring visual teaching strategies for enhancing mathematical thinking in grade 11 probability classes.

Lavinia Tangi- Jehova Nghidinwa from the Department of **Education**, Rhodes University has requested my permission to allow my child/ ward to participate in the above-mentioned research project.
The nature and the purpose of the research project, and of this informed consent declaration have been explained to me in a language that I understand.

I am aware that:

1. The purpose of the research project is to determine how the use of visualization tools in the teaching of probability can promote mathematical thinking at the secondary level.

2. The Rhodes University has given ethical clearance to this research project and I have seen/ may request to see the clearance certificate. **[Certificate number]**

3. By participating in this research project my child/ward will be contributing towards the improvement in the teaching of probability using different visualisation tools. Visualisation refers to the use of visual models to improve the teaching and learning of different concepts in mathematics, including probability. Since learners struggle to understand probability, exploring different teaching strategies could be beneficial for your child.

4. My child/ward will participate in the project by being present in the class where videos of their mathematics teacher will be recorded.

5. My child's participation is entirely voluntary and if my child/ward is older than seven (7) years, s/he must also agree to participate.

6. Should I or my child/ward at any stage wish to withdraw my child from participating further, we may do so without any negative consequences.

7. My child may be asked to withdraw from the research before it has finished if the researcher or any other appropriate person feels it is in my child's best interests, or if my child does not follow instructions.

8. Neither my child nor I will be compensated for participating in the research.

9. There may be risks associated with my child's participation in the project.

I am aware that

a. the following risks are associated with participation: All efforts will thus be made not to record and include the faces of any child in the class.
b. the following steps have been taken to prevent the risks: In the event of the face of a child being apparent, every effort will be made to blur the face on the video clip so as not to reveal the identity of that child. If this is not possible I would like to assure you that the visuals and videos where your child appears will only be used for the purpose of analysis of this study.

c. there is a 40% chance of the risk materialising

The researcher intends to publish the research results in the form of a thesis. The data collected will be handled with care and they will not be published without your consent. Therefore, data will be stored for 5 years in an external drive and clouds that will be locked (password protected) in a way that is hard to access. A back up copy will be saved on google account.
However, confidentiality and anonymity of records will be maintained and that my or my child's/ward's name and identity will not be revealed to anyone who has not been involved in the conduct of the research.

10. I will receive feedback in the form of thesis, workshops and journal publishing regarding the results obtained during the study.

11. This study is being done under the auspices of the Mathematics chair.

Any further questions that I might have concerning the research or my participation will be answered by Ms. Lavinia Tangi – Jehova Nghidinwa at **081 377 6668** or **Dr. Clemence Chikiwa** (supervisor), **Tell: +27 46 603 7210**, Email: c.chikiwa@ru.ac.za

12. By signing this informed consent declaration, I am not waiving any legal claims, rights or remedies that I or my child/ward may have.

13. A copy of this informed consent declaration will be given to me, and the original will be kept on record.

I, ..[redacted]............... have read the above information / confirm that the above information has been explained to me in a language that I understand and I am aware of this document's contents. I have asked all the questions that I wished to ask and these have been answered to my satisfaction. I fully understand what is expected of my child during the research.

I have not been pressurised in any way to let my child take part. By signing below, I voluntarily agree that my child/ward [redacted]..... (**insert name of child**), who is ....16.......... years old, may participate in the above-mentioned research project.

.... 01..June..20.....
Parent/Guardian's signature Witness **Date**

APPENDIX G: SURVEY QUESTIONNAIRE

RHODES UNIVERSITY
Where leaders learn

Survey questionnaire for mathematics teachers in the Khomas region.

Dear participants

You are invited to participate in a survey that wishes to establish the extent to which mathematics teachers in the Khomas region use visual tools to teach probability to promote mathematical thinking. This survey is part of an educational research project that will be carried out by Lavinia Tangi-Jehova Nghidinwa. Lavinia is a master's student at Rhodes University in Grahamstown, South Africa.

Your participation in this survey is voluntary. If you choose to participate, your personal information will remain strictly anonymous. Information that could be used to identify you or connect you to individual results will not be shared with staff in your school, region or country. Individual respondents will never be identified in any reports of results. The questionnaire poses no risk to you and there is no penalty for refusal to participate. You may withdraw from the study simply by returning the questionnaire without completing it, without any penalty.

It is important that you answer each question carefully so that the information provided reflects your situation as accurately as possible. It is estimated that it will require approximately 20 minutes to complete this questionnaire.

Your cooperation in completing this questionnaire is greatly appreciated.

Please note that there are no "right" or "wrong" answers to any of the questions. Once you have completed the questionnaire, place it into the return envelope provided and return it: Lavinia T Nghidinwa, contact details: 081 377 6668, Acacia high school, circuit 3, Khomasdal, Windhoek.

Again, thank you for your time, effort and thought in completing this questionnaire.

Section A
General information

Name ……………………………………………………………………………………………………

School ………………………………………………………………………………………………….

1. By the end of this school year, how many years will you have been teaching altogether? Please round to the nearest whole number ………………………………………………….

2. (a) What teacher training qualification do you have? Tick the appropriate column.

Grade 12	Certificate/diploma in education	Undergraduate degree in education	Honors degree in education	Masters of education

(b) In your teacher training did you cover the topic and the role of visualisation in mathematics? Please elaborate.

………
………
………
………………………………………………………………………………………………………

(c) How would you illustrate the concept of probability in a visual manner?

………
………
………
……………………………………………………………………………………………………

Section B

In this section, the questions refer to your mathematics teaching of probability. Please remember that this study's focus is on how visual tools can be used in the teaching of probability to promote mathematical thinking in a grade 11 mathematics class. [**Visual tools refers to the manipulatives that a teacher can use to teach probability**]

1. How do you teach probability in your classroom?

………
……………………………………………………………………

2. What visual materials do you use when teaching probability?

………
…………………………………………………………………

3. Where do you get these teaching materials and who prepares them for you?

………
……………………………………………………………………

4. In your view, can the use of visual tools help learners to understand the topic of probability?

………
……………………………………………………………………

5. How do the used visual tools (if there is any) enhance mathematical thinking?

………
…………………………………………………………………

6. Explain the importance of using visual materials in the teaching of probability, if there is any.

7. How best do you think teachers can improve the teaching of probability, if and only if there is need to do so?

……
…………………………………………………………………………

8. How do you make connections (apply probability in reality) or link probability to real-life phenomena or daily arithmetic?

……
…………………………………………………………………………

9. In your mathematics lessons that focus on probability, how often do you use visual materials to do the following:

Tick one column in each row	**Never**	**Some lessons**	**Most lessons**	**Every lesson**
Explain the reasoning behind an idea				
Represent and analyse relationships				
Work on problems for which there is no immediately obvious method of solution				
To solve exercises or problems				
To show equivalence				

10. In your probability teaching, when does the use of visual materials work best?

Tick one column in each row	**Never**	**Some lessons**	**Most lessons**	**Every lesson**
When learners work individually without assistance from the teacher.				
When learners work individually with assistance from the teacher.				
When learners work together as a class with the teacher teaching the whole class.				
When learners work together as a class with learners responding to one another				
When learners work in pairs or small groups without assistance from the teacher				
A Teacher does not use the visual tools at all				

The end

Thank you for the thought, time and effort you have put into completing this questionnaire.

APPENDIX H: OBSERVATION ANALYTICAL TOOL A

Observation schedule and analytical framework [thematic analysis]

Name of participant...

Date...

Topic..

Teaching strategies (Pedagogical aspects)	Codes	Definitions:	Descriptions:
Identify random events in nature, technology and society (IRE)	IRE	Randomness is a foundational concept in probability.	**IRE 1:** This may include random processes or experiments that show various interpretations and misconceptions held by learners. **IRE 2:** Emphasize the need to reinforce understanding of randomness in learners.
Analyze conditions of random events and derive appropriate modeling assumptions (ACR)	ACR	Refers to learners growing awareness of themselves as legitimate creators of mathematical knowledge. The teacher should be able to facilitate the probability lesson, by allowing learners to be actively involved in the discussion and solve challenging probability problems.	**ACR 1:** Learners to work in groups to set up experiments and the teacher use different levels of scaffolding and constraints to teach a problem. **ACR 2:** Developing learners' mathematical thinking rather than their memory skills. **ACR 3:** Intent to shape probability ideas and make connections in a coherent and sensible fashion. **ACR 4:** Highlight the importance of shaping learners' higher-level thinking by fostering learners' involvement in taking and defending their solutions against the claims of other learners.
Construct mathematical models for stochastic situations and explore various scenarios and outcomes from these models (CMM):	CMM	Observed when the teacher encouraging learners to keep working or thinking about a probability problem, gave them instrumental help and visuals tools that facilitate their progress, allowing plenty of time for learners to complete	**CMM 1:** Teachers use learners' inadequate solutions and mistakes to enhance instruction. **CMM 2:** Teacher comments on the problem-solving process or the strategies employed by the learners to eradicate identified mistakes in probability concepts like Probability of a single event, The probability scale from 0 to 1, Addition and multiplication of probabilities,

		probability tasks require learners to go back and try again when they reach inadequate solutions or encourage learners to come up with multiple strategies to tackle the tasks.	Application of probability, simple and combined events as well as Possibility diagrams; Tree diagrams and Venn diagrams.
Apply mathematical methods and procedures of probability (AMM).	AMM	The teacher should be able to administer practical investigations and projects to learners and let the learners present their work in class.	**AMM 1:** Daily practice and rituals of the classroom play an important part in how learners perceive and learn probability concepts. **AMM 2:** Learners create "insider knowledge" of probability behavior. **AMM 3:**Learners create discourse in probability from the norms associated with daily practices.
Verbal and nonverbal codes (VNVC) (Paivio, 2006)	VNVC	Refers to the use of both visual and nonvisual to enhance quality teaching and thinking in probability classroom.	**VNVC 1:** A fair integration of both visual and verbal codes in the presentation of probability lessons and assessment activities. **VNVC 2:** Teaching supports and builds on learners' intuitive strategies and these are used as a basis for the development of written forms of probability that accord with learners' verbally based and pictorially-based strategies. **VNVC 3:** Interchangeably uses the verbal and non-verbal codes to enhance learners' mathematical thinking in probability. **VNVC 4:** Assessment probability tasks are comprised of both symbolic notation and visual representations.

Adapted from Batanero et.al (2016) Probabilistic reasoning and Paivio (2006) Dual coding theory.

APPENDIX H: MATHEMATICAL THINKING ANALYTICAL TOOL B

An analytical framework for mathematical thinking. [Thematic analysis]

Mathematical Thinking Practices	**Code**	**Definition:**	**Description:**
Taking learners' ideas seriously (TLIS)	**TLIS**	Achieved when teachers value and build upon learners' intuitive understanding of probability concepts.	**TLIS1**: The teacher connects learners thinking to a more efficiently and abstract method by eliciting and valuing learners' initial solution strategies. **TLIS2**: Learners solve an unfamiliar yet meaningful probability problem, drawing from their prior knowledge & experience.
Encouraging multiple strategies and models (EMMS)	**EMMS**	Achieved when learners are able to generate, evaluate and utilize different probability strategies and models as well as recognize many ways to solve & represent solutions.	**EMMS1**: Teachers select relevant visuals that are aligned to the probability lesson objectives. **EMMS2**: Employs at least 2 probability models. EMMS3: Able to interchangeably select probability concepts using both visual and verbal codes. **EMMS4**: Teachers help to develop learners' mathematical thinking of various models & approaches to solving problems. **EMMS5**: Highlight different aspects of probability by using different strategies and models. **EMMS6**: Examining the same probability problem through different lenses deepens learners' overall understanding of the topic.
Pressing Learners Conceptually (PSC)	**PSC**	Achieved when learners Build connections between probability strategies and models and progressively formalizes those ideas and methods for solving problems.	**PSC1**: Teachers should give chance to learners to work on their own solution methods. **PSC2**: Teachers press the learners to connect and compare probability methods, generalize to new probability concepts, and relate to formal mathematical terms and conventions. **PSC3**: Learners move from their own informal methods to more formal and efficiently probability strategies. **PSC4**: Poses questions or assessment activities that encourage creative thought and innovative or mathematical thinking.
Addressing misconceptions (AM)	**AM**	It involves using learners' mistakes and misconceptions as valuable tools to build mathematical thinking.	**AM1**: Identify why and how probability misconceptions developed. **AM2**: Address probability misconceptions through models and discussion.

			AM3: Teachers take mistakes as opportunities for learners to engage in probability justification, evaluation, and inquiry. **AM4**: The teacher moves learners to a deeper level of probability understanding that precludes mistakes.
Focusing on the structure of mathematics (probability) (FSM)	**FSM**	Involves facilitating learners' understanding of fundamental, or structural, probability concepts.	**FSM1**: Teacher tie different probability concepts together, both within and across grade levels. **FSM2**: Instructions are focused on the structure of probability.

Adapted from Carney et al. (2014) Observation instrument for Developing Mathematical Thinking (DMT)

APPENDIX I: STIMULUS-RECALL INTERVIEW QUESTIONS

Stimulus- recall interviews questions

1. What informs your selection /preparation of visuals to teach the following probability concepts:

a) Introduction to probability

b) Independent event – how to draw a possibility space and tree diagram

c) Dependent event – how to calculate the probability of a simple combined event by using the possibility space and tree diagram.

2. During our lessons, you have been teaching learners how to work out probability using visual tools. In your view and experience do you think it is important to use visual tools when teaching probability? Why, please elaborate.

3. Will you continue using visual tools to teach probability? Motivate your answer.

4. Looking at the visual tools that you used in class, which visual tool, in particular, do you think can help learners to understand the content you taught better?

5. What experience did you gain working with visual tools that may enhance the teaching of probability to promote mathematical thinking as a result of participating in the intervention program?

6. What challenges did you experience using these visual tools? How can these challenges be overcome?

7. Has your participation in the intervention programme changed your way of teaching probability? Please explain.

APPENDIX J: LESSON PLAN WITH THREE PRESENTATIONS

	Lesson Plan	
Mathematics Research		**Subject: Mathematics**
Grade: 11	**Topic: Probability**	**Duration: 4 months**

1. **Teaching Resource and material used:**
   Mathematics y = mx + C textbook by Karen D'Emiljo and Excellent Mathematics by Jesaya Hambata & Gert van der Westhuizen. Presenter note on chalkboard, worksheets, visual tools: Play cards, Dice and coloured balls, coloured cubes and deck of cards.

2. **Lessons Objectives: By the end of the lesson, learners should be able to:**
   - Understand probability in practice, e.g. relative frequency.
   - Calculate the probability of simple combined events, using probability diagrams and tree diagrams where appropriate (in possibility diagrams outcomes will be represented by points on a grid and in tree diagrams, outcomes will be written at the end of branches and probabilities by the side of the branches).

3. **Lesson presentation:**
   - Introduction: State the topic (probability) and give the learners the focus of the lesson in terms of expectation: the learning objectives of the lesson.
   - Motivate learners to realise the importance of thinking mathematically, listening and active participation, doing activities (practice) as well as questioning.

Presentation of subject matter and learning activities

First Presentation: Probability of a single events and the probability scale

- Probability of a single event as either a fraction or a decimal (not ratio).
- The probability scale from 0 to 1.
  Explain how to calculate the probability: **P = $\dfrac{\text{Number of successful outcomes}}{\text{Total number of possible outcomes}}$**
- Interpret the probability of an event occurring as [1 – the probability of event not occurring]
- Recap the basic core content: Probability of simple and combined events, Possibility diagrams, Tree diagrams and Venn diagrams.

Second Presentation: Probability of simple combined events

Independent events: Calculate the probability of simple combined events

- Explain how to draw possibility diagrams (contingency tables) and tree diagrams.
- Explain how to calculate probability of simple combined events by using the possibility space and tree diagrams and Venn diagrams where appropriate.

Third Presentation: Teaching probability in practice

Applying the probability in practice such as using: explain probability by using scenarios

- Use any visual materials to apply the games approach
- Link games and scenarios to contingency tables, Venn diagrams and Tree diagrams to solve problems and/or to prove a point.

Learning activities during presentation: Practical activities in all lessons:

- Listen and take notes
- Do activities either individually or in groups using visual tools.
- Ask questions
- NB: Teacher facilitates using the mathematical models and ask the questions.
- Teacher facilitate the activities and the application of manipulative materials such as play cards, dice, coloured cubes and balls to specific activities.

4. **Consolidation:**
   - Emphasis based on the objective at the end of each presentation.
   - Conclude by summarizing the main parts to consider when answering questions i.e formulae.
5. **Assessment as class activity, or home work.**
   - Learners expose to various activities during lessons and home work (practice exercise question in $y = mx + c$).
   - Activity used during probability in practice lesson.
   - Do follow up of the previous work.
6. **Reflections:**
   - After repeating some of the challenging parts such as tree diagram of dependent events and monitoring homework before each lesson learners seemed to have understood the topic well.

Presenter: **Signature:**

APPENDIX K: CERTIFICATE OF EDITING

proofEDit

make sure it's correct

Certificate of editing

20 July 2021

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